Almost partitioning the hypercube into copies of a graph

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May 20, 2018

Abstract

Let H be an induced subgraph of the hypercube Q_k, for some k. We show that for some $c = c(H)$, the vertices of Q_n can be partitioned into induced copies of H and a remainder of at most $O(n^c)$ vertices. We also show that the error term cannot be replaced by anything smaller than $\log n$.

1 introduction

Given graphs G and H, an *H-packing of G* is a collection of vertex-disjoint copies of H in G. A *perfect H-packing* (also known as an H-factor) is an H-packing that covers all the vertices of the ground graph G (so, in order for G to have a perfect H-packing, $|H|$ must divide $|G|$). A natural question asks for conditions on G that imply the existence of an H-factor. For example, a well researched question asks for the smallest minimum degree that implies the existence of an H-factor. If H is an edge (and more generally if H is a path), then, by Dirac's theorem [4], if G has n vertices and minimum degree at least $n/2$ (and $|H|$ divides $|G|$), then G has a perfect H-packing. Corrádi and Hajnal [3] showed that $\delta(G) \geq 2n/3$ guarantees the existence of a perfect K_3-packing and Hajnal and Szemerédi [8] extended this result by showing that if $\delta(G) \geq (1 - 1/r)n$ then G has a perfect K_r-packing. We remark that these conditions on $\delta(G)$ are best possible.

After a series of papers by Alon and Yuster [1, 2] and by Komlós, Sárközy and Szemerédi [9], Kuhn and Osthus [10] found the smallest minimum degree condition that guarantees the existence of an H-factor, up to an additive constant error term, and for all H.

We consider a different problem, where instead of looking for H-packings in graphs of large minimum degree, we focus on H-packings of the hypercube Q_n. There are two obvious conditions for the

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existence of a perfect H-packing in Q_n: H has to be a subgraph of Q_n; and the order of H has to be a power of 2. Gruslys [5] showed that these two conditions are sufficient for large n, thus confirming a conjecture of Offner [12]. In fact, he showed that if H is an induced subgraph of Q_k for some k and $|H|$ is a power of 2, then there is a perfect packing of G into *induced* copies of H.

A similar problem concerns packings of the Boolean lattice $2^{[n]}$ into induced copies of a poset P. Note that here, if we drop the induced condition, we reduce to the case where P is a chain, thus in the case of posets we only consider induced copies of P. Again, there are two obvious necessary conditions: P must have a minimum and maximum elements; and the order of P has to be a power of 2. Lonc [11] conjectured that for large enough n, these conditions are also sufficient, and verified the conjecture for the case where P is a chain. This conjecture was recently solved by Gruslys, Leader and Tomon [7].

It is natural to ask what can be said when the divisibility condition does not hold. Gruslys, Leader and Tomon [7] conjectured that if P is a poset with a maximum and a minimum, then there is a P-packing of Q_n that covers all but at most c elements, where $c = c(P)$ is a constant that depends on P. This conjecture was recently proved by Tomon [14].

In light of this result, it is natural to ask if a similar phenomenon holds in the case of H-packings of the hypercube Q_n. Namely, if H is a subgraph of Q_k for some k, how large an H-packing of Q_n can we find? As our first main result, we show that if H is a subgraph of Q_k then there is an H-packing of Q_n that covers all but at most $O(n^c)$ vertices, where $c = c(H)$.

Theorem 1. *Let H be an induced subgraph of Q_k for some k. Then there exists a packing of G by induced copies of H, such that at most $O(n^c)$ vertices remain uncovered, where $c = c(H)$.*

It is natural to wonder if the number of uncovered vertices can be reduced to be at most $c(H)$. Perhaps surprisingly, it turns out that this is not always the case. As our second main result, we show that a $(P_3)^3$-packing of Q_n misses at least $\log n$ vertices (P_3 is the path on three vertices).

Theorem 2. *In every $(P_3)^3$-packing of Q_n, at least $\log n$ points are uncovered.*

1.1   Notation

We denote the path on l vertices by P_l. When we say that G can be partitioned into copies of H, we mean that there exists a perfect H-packing of G. For graphs G_1 and G_2, we denote by $G_1 \times G_2$ the Cartesian product of G_1 and G_2, which has vertex set $V(G_1) \times V(G_2)$ and $((u_1, v_1), (u_2, v_2))$ is an edge iff $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. The n-th power G^n of G is defined to be $G \times \ldots \times G$ (where G appears n times).

1.2 Structure of the paper

This paper consists of three parts. In the first part (see Section 3) we prove that if H is an induced subgraph of Q_k for some k, then for sufficiently large n, there is a perfect packing of $(P_{2|H|})^n$ into induced copies of H (see Theorem 3). In the second part (Section 4), we prove that there is a packing of Q_n by induced copies of $(P_l)^t$ which leaves at most $O(n^{t-1})$ vertices uncovered. These two parts easily combine to form a proof of Theorem 1. Finally, in the third part (Section 5) we prove Theorem 2, thus showing that the error term $O(n^{t-1})$ cannot be replaced by something smaller than $\log n$.

Before proceeding to the proofs, we give an overview of them in Section 2. We finish the paper with concluding remarks and open problems in Section 6.

2 Overview of the proofs

In this section we give an overview of the proofs in this paper.

2.1 Partitioning $(P_{2l})^n$

Our first aim in this paper is to prove Theorem 3.

Theorem 3. *Let H be an induced subgraph of Q_k for some k. Then $(P_{2|H|})^n$ can be partitioned into induced copies of H, whenever n is sufficiently large.*

Our proof follows the footsteps of Gruslys [5] who proved that if H is an induced subgraph of a hypercube whose order is a power of 2, then for large n there is a perfect packing of Q_n into induced copies of H.

An important tool in the proof of Theorem 3 is a result of Gruslys, Leader and Tomon [7] (introduced by Gruslys, Leader and Tan [6] for tiling of $\mathbb{Z}^n$) which gives a a general method for proving the existence of perfect packings of a product space A^n into copies of a subset S of A. Given a subset S of A, a collection of copies of S in A^n (which may contain a certain copy several times) is called an l-partition (($r \mod l$)-partition) if every vertex in A^n is covered by exactly l (($r \mod l$)) copies of S. We note that a 1-partition is simply a perfect packing. Trivially, the existence of a perfect packing implies the existence of an l-partition and a ($1 \mod l$)-partition. Remarkably, the aforementioned result of Gruslys, Leader and Tomon shows that, roughly speaking, the opposite is true. Namely, they showed that if there exists l for which A^m admits an l-partition and a ($1 \mod l$)-partition into copies of S, then, for large n, A_n admits a perfect packing into copies of S. The precise statement of this result is given in Theorem 5.

The existence of an $|H|$-partition of $(P_{2|H|})^n$ into induced copies of H is a simple observation (see Observation 6). The existence of a $(1 \mod |H|)$-partition of $(P_{2|H|})^n$ into copies of H is more difficult to prove, but it is quite straightforward to adapt the methods of Gruslys [5] to work in our setting. These two facts, together the aforementioned result [7], form the proof of Theorem 3.

2.2 Almost partitioning Q_n into powers of a path

Our second aim is to prove Theorem 4.

Theorem 4. *For any l and t, there is a packing of Q_n into induced copies of $(P_l)^t$, for which at most $O(n^{t-1})$ vertices are uncovered.*

The fact that Q_n is Hamiltonian shows that there is a P_l-packing of Q_n missing fewer than l vertices. In fact, if l divides $2^n - 1$, then exactly one vertex remains uncovered. This observation allows us to prove the existence of a $(P_l)^t$-packing of Q_n with at most $O(n^{t-1})$ uncovered vertices, whenever l is odd (see Observation 8). It is then not hard to conclude that the same holds for all l (see Corollary 9) using the observation that $(P_{2l})^t$ is a subgraph of $(P_l)^t \times Q_t$.

Note that this does not imply Theorem 4, since we require that the copies of $(P_l)^t$ are induced. We notice that if H is a graph on l vertices with a Hamilton path, then $H \times P_{l-1}$ has a perfect packing into induced P_l's (see Observation 11). This fact, with a little more work, allows us to use the packing of Q_n into (not necessarily induced) copies of $(P_{l'})^t$ to obtain a packing into induced copies of $(P_l)^t$ (where l' is suitably chosen).

We note that Theorem 1 follows from Theorems 3 and 4.

Proof of Theorem 1. Let H be a subgraph of Q_k. Then by Theorem 3, there exists m for which there is a perfect packing of $(P_{2|H|})^m$ into induced copies of H. By Theorem 4, there is a packing of Q_n into induced copies of $(P_{2|H|})^m$, such that at most $O(n^{m-1})$ vertices are uncovered. Hence there exists a packing of Q_n into induced copies of H with at most $O(n^{m-1})$ uncovered vertices (note that m depends only on H). $\qquad\square$

2.3 Lower bound on the number of uncovered vertices

Our final aim is to prove Theorem 2.

Theorem 2. *In every $(P_3)^3$-packing of Q_n, at least $\log n$ points are uncovered.*

We use the properties of Q_n and of $(P_3)^3$ to conclude that the size of the intersection of any co-dimension-2 subcube of Q_n with any copy of $(P_3)^3$ is divisible by 3. In fact, we deduce this from a similar statement for $(P_3)^t$ (see Proposition 13). We conclude that the set of uncovered vertices in a $(P_3)^3$-packing of Q_n forms a *separating family* for $[n]$, implying that it has size at least $\log n$.

3   Perfect H-packings of $(P_{2|H|})^n$

Our main aim in this section is to prove Theorem 3.

Theorem 3. *Let H be an induced subgraph of Q_k for some k. Then $(P_{2|H|})^n$ can be partitioned into induced copies of H, whenever n is sufficiently large.*

Recall that a result of Gruslys, Leader and Tomon [7] implies that it suffices to find l- and $(1 \bmod l)$-partitions into copies of H. Before stating their result precisely, we introduce some notation.

Let A be a set. We identify $A^n \times A^m$ with A^{n+m} (whenever m and n are positive integers). Thus, for any $x \in A^n$ and $y \in A^m$, we treat (x, y) as an element of A^{n+m}. Given a set X in A^n, and a permutation $\pi : [n] \to [n]$, we define $\pi(X)$ to be the image of X under the permutation of the coordinates according to π. In other words, $\pi(X) = \{(x_{\pi(1)}, \ldots, x_{\pi(n)}) : (x_1, \ldots, x_n) \in X\}$. Finally, given sets X in A^m and Y in A^n where $m \leq n$, we say that Y is a *copy* of X if $Y = \pi(X \times \{y\})$ for some $y \in A^{n-m}$.

Theorem 5 (Gruslys, Leader, Tomon [7]). *Let $\mathcal{F}$ be a family of subsets of a finite set A. If there exists l for which $\mathcal{F}$ contains an l-partition and a $(1 \bmod l)$-partition of S, then there exists n for which S^n admits a partition into copies of elements in $\mathcal{F}$.*

The task of finding an l-partition is quite simple. In fact, it follows directly from the analogous result in [5] and the fact that $(P_{2l})^n$ can be partitioned into copies of Q_n. For the sake of completeness, we include the proof here.

Observation 6. *Let H be an induced subgraph of Q_k for some k. Then there is an $|H|$-partition of $(P_{2|H|})^n$ into induced copies of H, for any $n \geq k$.*

Proof. Denote $l = |H|$. Note that, since the path P_{2l} can be partitioned into l edges, its n-th power $(P_{2l})^n$ can be partitioned into l^n induced copies of Q_n. Thus, it suffices to exhibit an l-partition of Q_n into induced copies of H. Let X be the vertex set of some induced copy of H in Q_n. We consider the set of all *shifts* of X. For every $u \in Q_n$, we note that the set $X + u = \{x + u : x \in X\}$ (addition is done coordinate-wise and modulo 2) is an induced copy of H in Q_n. Consider the collection $\{X + u : u \in Q_n\}$. By symmetry, every vertex in Q_n is covered by the same number of sets. Furthermore, there are 2^n such sets, each covers l points, so the number of times each vertex is covered is $\frac{2^n l}{2^n} = l$. So, we found an l-partition of Q_n into induced copies of H. $\qquad\square$

The next task, of finding a $(1 \bmod l)$-partition of $(P_{2l})^n$ into copies of H, is significantly harder. Unlike Observation 6, we cannot directly apply the analogous result of Gruslys [5]. Instead, we adapt his method to our setting.

Theorem 7. *Let H be a non-empty induced subgraph of Q_k for some k. Then there is a $(1 \mod l)$-partition of $(P_{2l})^k$ into induced copies of H.*

We note that in Theorem 3, unlike Observation 6, there is no restriction on the order of H.

Before proceeding to the proof of Theorem 7, we show how to prove Theorem 3 using Observation 6 and theorem 7.

Proof of Theorem 3. Denote $l = |H|$. Note that it suffices to show that for some n the graph $(P_{2l})^n$ can be partitioned into induced copies of H. Recall that H is an induced subgraph of Q_k, for some k. By Theorem 7, there is a $(1 \mod l)$-partition of $A = (P_{2l})^k$ into induced copies of H. By Observation 6 there is an l-partition of A into induced copies of H. Hence, by Theorem 5 there exists n for which there is a perfect H-packing of $A^n = (P_{2l})^{kn}$, as required. $\qquad\square$

We now proceed to the proof of Theorem 7.

Proof of Theorem 7. We shall prove the following slightly stronger claim: if H is a non-empty induced subgraph of Q_k, then there is a $(1 \mod r)$ partition of $(P_{2l})^k$ into isometric copies of H.

Let us first explain briefly what we mean by an isometric copy of H in a graph G. We consider the graphs Q_k and G together with the metric coming from the graph distance. An isometric copy of H is the image of H under an isometry $f : Q_k \to G$ (here we fix a particular embedding of H in Q_k).

Define H_- and H_+ as follows.

$$H_- = \{u \in Q_{k-1} : (u, 0) \in H\}$$
$$H_+ = \{u \in Q_{k-1} : (u, 1) \in H\}.$$

We prove the claim by induction on k. It is trivial for $k = 1$ (then H is either a single vertex or an edge), so suppose that $k \geq 2$ and the claim holds for $k - 1$. Note that we may assume that H_- and H_+ are both non-empty. We shall show that $(P_{2l})^k$ has a $(1 \mod l)$-partition into isometric copies of H.

We denote the vertices of P_{2l} by $\{0, 1, \ldots, 2l-1\}$. Let $p \in [2l-2]$. By induction, there is a collection of isometric copies of H_- in $(P_{2l})^{k-1} \times \{p\}$ such that each point is covered $(1 \mod l)$ times. Let A be the vertex set of such a copy of H_-. Then there is an isometric copy of H in $(P_{2l})^{k-1} \times \{p, p+1\}$ whose intersection with $(P_{2l})^{k-1} \times \{p\}$ is A. It follows that there exists a collection $\mathcal{H}$ of isometric copies of H in $(P_{2l})^{k-1} \times \{p, p+1\}$ for which every point in $(P_{2l})^{k-1} \times \{p\}$ is covered $(1 \mod l)$ times. Let $\mathcal{H}'$ be the collection of isometric copies of H in $(P_{2l})^{k-1} \times \{p-1, p\}$, which is the image

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of $\mathcal{H}$ under the map from $(P_{2l})^{k-1} \times \{p, p+1\}$ to $(P_{2l})^{k-1} \times \{p-1, p\}$ obtained by changed the last coordinate from $p+1$ to $p-1$.

Denote by $\mathcal{H}_p$ the collection $(l-1)\mathcal{H} + \mathcal{H}'$ (i.e. each copy of H in $\mathcal{H}$ is taken $l-1$ times). $\mathcal{H}$ is a collection of copies of H in $(P_{2l})^{k-1} \times \{p-1, p, p+1\}$ which we view as a collection of copies of H in $(P_{2l})^k$. For every $x \in (P_{2l})^k$, the number of times x is covered is

$$
w_p(x) = \begin{cases}
(1 \mod l) & x \in (P_{2l})^{k-1} \times \{p+1\} \\
(-1 \mod l) & x \in (P_{2l})^{k-1} \times \{p-1\} \\
(0 \mod l) & \text{otherwise}
\end{cases}
$$

Let $\mathcal{G}$ be the collection of isometric copies of H obtained by taking i copies of $\mathcal{H}_{2i}$ and $\mathcal{H}_{2i-1}$ for each $i \in [l-1]$. We show that every vertex in $(P_{2l})^n$ is covered $(1 \mod l)$ time by $\mathcal{G}$. Let $x \in (P_{2l})^{k-1}$. Then $(x, 0)$ and $(x, 1)$ are covered $(1 \mod l)$ times (they get non zero weight only in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively). The vertices $(x, 2i)$ and $(x, 2i+1)$ (where $i \in [l-2]$) are covered $(-i \mod l)$ times by $\mathcal{H}_{2i-1}$ and $\mathcal{H}_{2i}$ respectively, and $(i+1 \mod l)$ times by $\mathcal{H}_{2i+1}$ and $\mathcal{H}_{2i+2}$, so in total they are covered $(1 \mod l)$ times. Finally, $(x, 2l-2)$ and $(x, 2l-1)$ are covered $l-1$ times by $\mathcal{H}_{2l-3}$ and $\mathcal{H}_{2l-2}$ respectively, so the number of times they are covered is $(-(l-1) \mod l) = (1 \mod l)$. □

4 Almost partitioning the hypercube into powers of a path

Our main aim in this section is to prove Theorem 4.

Theorem 4. *For any l and t, there is a packing of Q_n into induced copies of $(P_l)^t$, for which at most $O(n^{t-1})$ vertices are uncovered.*

Before proceeding to the proof, we make several observations and mention a result that we shall use. The following observation makes use of the fact that Q_n is Hamiltonian to prove the special case of Theorem 4 where l is odd and the requirement that the copies are induced is dropped.

Observation 8. *Let l be odd, and let t be a positive integer. Then there exists a $(P_l)^t$-packing of Q_n, that covers all but at most $O(n^{t-1})$ vertices.*

Proof. It is well known that Q_m is Hamiltonian. Therefore, if l divides $2^m - 1$, then Q_m may be partitioned into copies of P_l and a single vertex. Note that by the Fermat-Euler theorem, there exists m such that l divides $2^m - 1$ (take $m = \phi(l)$, where $\phi(n)$ is Euler's totient function, counting the number of integers $p < n$ for which p and n are coprime). It follows that all but at most $2^{m(t-1)} \cdot |[r]^{<t}|$ vertices of $(Q_m)^r$ can be partitioned into copies of $(P_l)^t$. Given any n, write $n = rm + a$ where $a < r$. Since we may view Q_n as $(Q_m)^r \times Q_a$, there is a collection of pairwise

disjoint induced copies of $(P_l)^t$ that covers all but at most $2^{a+m(t-1)} \cdot |[r]^{<t}| \leq 2^m \cdot n^{t-1} = O(n^{t-1})$ vertices. □

The following corollary allows us to extend Observation 8 to all l.

Corollary 9. *Let l and t be integers. Then all but $O(n^{t-1})$ vertices of Q_n may be partitioned into copies of $(P_l)^t$.*

Proof. We prove the statement by induction on i, the maximum power of 2 that divides l. If $i = 0$, l is odd, and the statement follows from Observation 8. Now suppose that $i \geq 1$. Write $l = 2k$ and $Q_n = Q_{n-t} \times Q_t$. By induction, there is a collection of pairwise disjoint copies of $(P_k)^t$ that covers all but at most $O(n^{t-1})$ vertices in Q_{n-t}. Note that the product $P_k \times Q_1$ is Hamiltonian, i.e. it spans a $P_{2k} = P_l$. We conclude that Q_n may be covered by pairwise disjoint copies of $(P_i)^t$ and a remainder of at most $O(n^{t-1} \cdot 2^t) = O(n^{t-1})$ vertices. □

Let $\mathcal{H}_l$ be the collection of graphs on l vertices which have a Hamilton path, and let $(\mathcal{H}_l)^t = \{H_1 \times \ldots \times H_t : H_i \in \mathcal{H}\}$. We note that the proofs of Observation 8 and Corollary 9 actually give the following slightly stronger statement.

Corollary 10. *Let l and t be integers. Then there is a collection of pairwise disjoint copies of graphs in $(\mathcal{H}_l)^t$ that covers all but at most $O(n^{t-1})$ vertices of Q_n.*

In order to obtain an almost-partition into induced copies of $(P_l)^t$, we need two more ingredients. One is the following observation.

Observation 11. *Let $H \in H_l$. Then $H \times P_{l-1}$ may be partitioned into induced copies of P_l.*

Proof. Denote the vertices of H by $[l]$ and suppose that $(1, \ldots, l)$ is a path in H. Similarly, we denote the vertices of P_{l-1} by $[l-1]$. Let Q_i be the path $((i, 1), \ldots, (i, l-i), (i+1, l-i), \ldots, (i+1, l-1))$, for $i \in [l-1]$ (see Figure 1). It is easy to see that each Q_i is an induced P_l. □

The second ingredient is a result of Ramras [13], which states that if $n + 1$ is a power of 2, then Q_n may be partitioned into antipodal paths. In particular, we have the following corollary.

Corollary 12 (Ramras [13]). *If $n + 1$ is a power of 2, then Q_n may be partitioned into induced copies of P_{n+1}.*

We are now ready to prove Theorem 4.
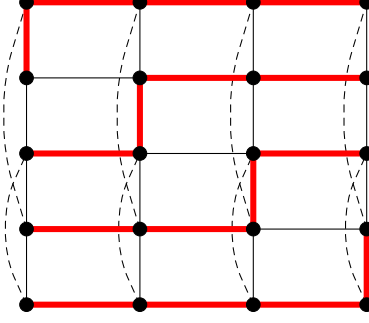
Figure 1: Illustration of the paths in Observation 11 (vertical lines denote copies of H)

Proof of Theorem 4. Let m be minimal such that $2^m \geq l^2$. Suppose that $2^m = a(\bmod\ l)$ where $0 \leq a < l$ and write $2^m = (l - 1 - a)(l - 1) + b$. Note that $(l - 1 - a)(l - 1) < l^2$ so $b > 0$, and by choice of m, $b \leq 2l^2$. Furthermore, $b = -1(\bmod\ l)$. The path P_{2^m} can be partitioned into $l - 1 - a$ copies of P_{l-1} and one copy of P_b. It follows from Corollary 12 that Q_{2^m-1} may be partitioned into induced copies of P_{l-1} and P_b, implying that $(Q_{2^m-1})^{2t}$ may be partitioned into induced copies of $(P_{l-1})^t$ and $(P_b)^t$.

Write $Q_n = Q_{n-t(2^m-1)} \times (Q_{2^m-1})^{2t}$. Recall that by Corollary 9, $Q_{n-t(2^m-1)}$ may be partitioned into copies of graphs in $(\mathcal{H}_{b+1})^t$ and a remainder of at most $O(n^{t-1})$ vertices.

Combining these two facts, we conclude that Q_n can be partitioned into copies of graphs isomorphic to $\prod_{i \in [t]}(H_i \times P_x)$ (where $H_i \in \mathcal{H}_{b+1}$ and $x \in \{l - 1, b\}$), and a remainder of at most $O\left(n^{t-1} \cdot 2^{t(2^m-1)}\right) = O(n^{t-1})$. We claim that if $H \in \mathcal{H}_{b+1}$ and $x \in \{l - 1, b\}$, then $H \times P_x$ may be partitioned into induced copies of P_l. Indeed, if $x = b$ then by Observation 11, $H \times P_x$ may be partitioned into induced P_{b+1}'s, which may in turn be partitioned into induced P_l's (since, by choice of b, l divides $b+1$). A similar argument holds if $x = l - 1$: first note that H may be partitioned into graphs in $\mathcal{H}_l$ (since l divides $b + 1$), and the product of these graphs with P_{l-1} may be partitioned into induced copies of P_l, by Observation 11. It follows that Q_n may be partitioned into induced copies of $(P_l)^t$ and a remainder of order at most $O(n^{t-1})$. □

5   A lower bound on the number of uncovered vertices

In this section we prove Theorem 2.

Theorem 2. *In every $(P_3)^3$-packing of Q_n, at least $\log n$ points are uncovered.*

We start by proving the following propositions that characterises the intersection of a copy of $(P_3)^t$ in Q_n with a subcube of co-dimension 1.

Proposition 13. *Let H be a copy of $(P_3)^k$ in Q_n. Then the intersection of H with any subcube S of co-dimension 1 is a copy of one of the following graphs: $\emptyset$, $(P_3)^{k-1}$, $P_2 \times (P_3)^{k-1}$ or $(P_3)^k$.*

Proof. Let S be the vertex set of a subcube of Q_n of co-dimension 1. Write $H = H' \times P_3$, where every H' is a copy of $(P_3)^{k-1}$, and denote $H_i = H' \times \{i\}$ (where $V(P_3) = \{1,2,3\}$). We prove the statement by induction on k.

Let $k = 1$, then each H_i is a single vertex. Without loss of generality, H_2 is in S (otherwise consider the complement of S). But then at least one of H_1 and H_3 also are in S (because every vertex in S has exactly one neighbour outside of S). So, without loss of generality, H_1 is in S. It follows that $V(H) \cap S$ is either H or $H_1 \times H_2$, as claimed.

Now suppose that $k \geq 2$. Then by induction, and without loss of generality, the intersection of S with H_1 is either H_1 or a copy of $P_2 \times (P_3)^{k-1}$.

Suppose that the first case holds, i.e. the intersection of S with H_1 is H_1. Then, if any vertex in H_2 is in S, all vertices of H_1 are in S (since every vertex in S has exactly one neighbour outside of S). In other words, H_2 is either contained in S or it is contained in $\bar{S}$, the complement of S. If the former holds, then, similarly, H_2 is contained in either S or $\bar{S}$, and if the latter holds then H_2 is contained in $\bar{S}$ (since every vertex in H_2 is in $\bar{S}$ and has a neighbour in $S \cap H_1$). It follows that the intersection of S with H in this case is H_1, $H_1 \times H_2$, or H, as required.

Now suppose that the second case holds, i.e. the intersection of H_1 with S is a copy of $P_2 \times (P_3)^{k-1}$. Then we may write $H_1 = H'' \times P_3$ where H'' is a copy of $(P_3)^{k-2}$, and $H'' \times \{1,2\}$ (where $V(P_3) = \{1,2,3\}$) is the intersection of H_1 with S. Denote $H_{i,j} = H'' \times \{i\} \times \{j\}$. So $H_{1,1}$ and $H_{1,2}$ are in S and $H_{1,3}$ is in $\bar{S}$. It follows that $H_{2,2}$ is in S (otherwise some vertex in $H_{1,2}$ would have two neighbours in $\bar{S}$); $H_{2,1}$ is in S (otherwise a vertex of $\bar{S} \cap H_{2,1}$ would have two neighbours in S); and $H_{2,3}$ is in $\bar{S}$. Similarly, $H_{3,1}$ and $H_{3,2}$ are contained in S and $H_{3,3}$ is in $\bar{S}$. It follows that the intersection of H with S is a copy of $P_2 \times (P_3)^{k-1}$. $\square$

We are now ready for the proof of Theorem 2

Proof of Theorem 2. Let $\mathcal{H}$ be a collection of pairwise disjoint copies of $(P_3)^3$ in Q_n and let S be a subcube of co-dimension 2 in Q_n, and let S' be a subcube of co-dimension 1 in Q_n that contains S. Let $H \in \mathcal{H}$. Then, by Proposition 13, the intersection of H with S' is either the empty set or it is the disjoint union of up to three copies of $(P_3)^2$. It follows from Proposition 13 that the intersection of H with S is the disjoint union of copies of P_3. In particular, since 3 does not divide the order of S, at least one vertex in S is not covered by $\mathcal{H}$.

Let $\mathcal{P}$ be the collection of subsets of $[n]$ that correspond to vertices of Q_n that are not covered by $\mathcal{H}$ (where we consider the usual map between Q_n and $\mathcal{P}([n])$ that sends a vertex u in Q_n to the set of

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elements in $[n]$ whose coordinates in u is 1). We claim that the collection $\mathcal{P}$ is a *separating family* for $[n]$, namely, for every distinct elements i and j in $[n]$, there is a set $A \in \mathcal{P}$ that contains i but not j. Indeed, given distinct i and j in $[n]$, let S be the subcube of co-dimension 2 of vertices whose i-th coordinate is 1 and whose j-th coordinate is 0. Then S contains a vertex which is uncovered by $\mathcal{H}$. This vertex corresponds to a set in $\mathcal{P}$ that contains i but not j. It is a well known fact that a family that separates $[n]$ has size at least $\log n$. It follows $\mathcal{P}$ has size at least $\log n$, implying that at least $\log n$ vertices of Q_n are not covered by $\mathcal{H}$. $\qquad\square$

We remark that by considering $(P_3)^{2k+1}$ packings of Q_n, the number of missing vertices can be shown to be at least $(1 + o(1))k \log n$ (since the subsets corresponding to the missing vertices form a separating system for $[n]^{(k)}$).

6 Concluding remarks

We showed that if H is an induced subgraph of Q_k then there exists a packing of Q_n into induced copies of H, which misses at most $O(n^c)$, for $c = c(H)$. On the other hand, we showed that the error term cannot be replaced by anything smaller than $\log n$ (or, as we remarked in Section 5 by $c \log n$ for any c). It would be very interesting to close the gap between the two bounds.

We believe that the upper bound, of $O(n^c)$ is closer to the truth, i.e., we believe that there exist graphs H for which at least $\Omega(n^c)$ vertices remain uncovered in any H-packing of Q_n. More specifically, it seems plausible to believe that every $(P_3)^k$ packing of Q_n leaves at least $\Omega(n^{k-1})$ vertices uncovered. We thus state the following question.

Question 14. *Is there a $(P_3)^k$ packing of Q_n for which the number of uncovered points is at most $o(n^{k-1})$?*

In this paper we are interested in H-packings of Q_n, which can be viewed as $(P_2)^n$. It would be interesting to consider the more general setting of H-packings of G^n. We mention a conjecture of Gruslys [5].

Conjecture 15 (Gruslys [5])**.** *Let G be a finite vertex-transitive graph, and let H be an induced subgraph of G. Suppose further that $|H|$ divides $|G|$. Then for some n there is a perfect H-packing of G^n.*

We note that the conjecture does not hold if we drop the vertex-transitivity (see Proposition 9 in [5]).

In Section 1, we mentioned a recent result of Tomon [14] who proved that if P is a poset with a minimum and a maximum, then the Boolean lattice $2^{[n]}$ can be partitioned into copies of P and a

remainder of at most c elements, where $c = c(P)$. It would be interesting to generalise his result to all posets P, dropping the requirement of the existence of a minimum and a maximum. This would resolve a conjecture of Gruslys, Leader and Tomon [7].

Conjecture 16 (Gruslys [7])**.** *Let P be a poset. Then the Boolean lattice $2^{[n]}$ can be partitioned into copies of P and a remainder of at most $c = c(P)$ elements.*

Finally, we mention a question about Hamilton paths of Q_n. Recall that in order to prove that Q_n can be almost partitioned into induced copies of $(P_l)^t$, we first proved this statement without requiring the copies to be induced. That followed easily from the fact that Q_n is Hamiltonian. We then used such a partition to obtain a partition of Q_n into induced copies of $(P_l)^t$. A more direct approach could be to find a Hamilton path P in Q_n for which every l consecutive vertices induced a P_l. We were unable to determine if such a Hamilton path exists. We thus conclude the paper with the following question.

Question 17. *Let l be integer. Is it true that for sufficiently large n, there is a Hamilton path Q_n for which every l consecutive vertices induce a P_l?*

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