On a problem of Brown, Erdős and Sós

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Abstract

Let $f^{(r)}(n; s, k)$ be the maximum number of edges in an n-vertex r-uniform hypergraph not containing a subhypergraph with k edges on at most s vertices. Recently, Delcourt and Postle, building on work of Glock, Joos, Kim, Kühn, Lichev and Pikhurko, proved that the limit $\lim_{n\to\infty} n^{-2} f^{(3)}(n; k+2, k)$ exists for all $k \geq 2$, solving an old problem of Brown, Erdős and Sós (1973). Meanwhile, Shangguan and Tamo asked the more general question of determining if the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$ exists for all $r > t \geq 2$ and $k \geq 2$.

Here we make progress on their question. For every even k, we determine the value of the limit when r is sufficiently large with respect to k and t. Moreover, we show that the limit exists for $k \in \{5, 7\}$ and all $r > t \geq 2$.

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1 Introduction

An (s, k)-configuration in an r-uniform hypergraph (henceforth *r-graph*) is a collection of k edges spanning at most s vertices. Brown, Erdős and Sós [1] started the investigation of the function $f^{(r)}(n; s, k)$, defined as the maximum number of edges in an n-vertex r-graph not containing an (s, k)-configuration. In particular, they showed that $f^{(r)}(n; s, k) = \Omega(n^{(rk-s)/(k-1)})$ for all $s > r \geq 2$ and $k \geq 2$. Suppose now that the exponent $t := (rk - s)/(k - 1)$ is an integer, so $s = k(r - t) + t$. Observe that s is the number of vertices spanned by a k-edge r-graph where the edges can be ordered so that all but the first edge share exactly t vertices with the previous edges. In particular, any set of vertices of size t which is contained in k distinct edges creates a $(k(r-t)+t, k)$-configuration. Therefore $f^{(r)}(n; k(r-t)+t, k) = O(n^t)$ and, with the above result of Brown, Erdős and Sós, we have

$$f^{(r)}(n; k(r-t)+t, k) = \Theta(n^t)\,.$$

A major open problem is the following conjecture, which was proposed by Shangguan and Tamo [8] and generalises an old conjecture of Brown, Erdős and Sós [1] (corresponding to $r = 3$ and $t = 2$).

Conjecture 1.1. *For any positive integers r, k, t, the limit*

$$\pi(r, t, k) := \lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$$

exists.

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We can assume $k \geq 2$ and $t \in [r-1]$, as otherwise Conjecture 1.1 is trivial. Moreover, when $t = 1$, it is easy to establish that the limit exists and that $\pi(r, 1, k) = \frac{k-1}{(k-1)(r-1)+1}$, as already observed in [4]. Indeed, the extremal constructions are vertex-disjoint unions of loose trees with $k-1$ edges, while the upper bound follows from the fact any collection of k edges, which can be ordered so that all but the first edge share at least one vertex with the previous ones, is a $(k(r-1)+1, k)$-configuration. Therefore we can in fact assume $k \geq 2$ and $t \in [2, r-1]$, and we will do so in the rest of the paper. Recently, significant progress has been made towards Conjecture 1.1 and we now summarise the main developments.

We start by discussing the results concerning the original conjecture of Brown, Erdős and Sós, that is Conjecture 1.1 for $r = 3$ and $t = 2$, i.e. the existence of $\pi(3, 2, k)$. Brown, Erdős and Sós studied the case $k = 2$ and showed [1] that the limit is $\pi(3, 2, 2) = 1/6$. More than 40 years later, Glock [3] proved the conjecture for $k = 3$ and determined that $\pi(3, 2, 3) = 1/5$. Very recently, Glock, Joos, Kim, Kühn, Lichev and Pikhurko [4] proved the conjecture for $k = 4$ and determined that $\pi(3, 2, 4) = 7/36$. In the concluding remarks, they also claim that their methods can be adapted to show that $\pi(3, 2, k) = 1/5$ for $k \in \{5, 7\}$. Finally, Delcourt and Postle [2] proved the Brown–Erdős–Sós conjecture in full, i.e. they showed that $\pi(3, 2, k)$ exists for all $k \geq 2$, although their method does not provide an explicit value for the limit.

Shanguann and Tamo [8] adapted the methods in [3] to any uniformity and showed that $\pi(r, 2, 3) = 1/(r^2 - r - 1)$, and Shanguann [7] adapted [2] to any uniformity and showed that $\pi(r, 2, k)$ exists (but provided no explicit value).

Concerning the general conjecture, the case $k = 2$ follows from the work of Rödl [6] on the existence of asymptotic Steiner systems, and we have $\pi(r, t, 2) = \frac{1}{t!}\binom{r}{t}^{-1}$. Glock, Joos, Kim, Kühn, Lichev and Pikhurko [4] settled the cases $k = 3$ and $k = 4$ by showing that $\pi(r, t, 3) = \frac{2}{t!}\left(2\binom{r}{t} - 1\right)^{-1}$ for every $r \geq 2$ and $\pi(r, t, 4) = \frac{1}{t!}\binom{r}{t}^{-1}$ for every $r \geq 4$ (note that the case $r = 3$ (and $t = 2$) is covered by one of the results mentioned above and does not follow the same pattern).

Our first result provides the exact value of the limit when k is even and r is sufficiently large in terms of k and t.

Theorem 1.2. *Let k be an even positive integer and $t \geq 2$ an integer. Then, for every integer r satisfying $r \geq t + (k^3 \cdot t!)^{1/t}$, we have that $\lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) = \frac{1}{t!}\binom{r}{t}^{-1}$.*

We remark that, as mentioned above, this was already known for $k = 2$ [6] and $k = 4$ [4] (and all r). Moreover, it is interesting to observe that the behaviour for odd k is potentially different. For example, from [4], it holds that $\pi(r, t, 3) = \frac{2}{t!}\left(2\binom{r}{t} - 1\right)^{-1}$. Therefore, we now focus on the case of k being odd.

Firstly, we completely settle Conjecture 1.1 for $k = 5$.

Theorem 1.3. *Let r, t be integers satisfying $r > t \geq 2$. Then the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 5(r-t)+t, 5)$ exists.*

Finally, we settle Conjecture 1.1 for $k = 7$. (Our proof does not work when $r = 3$ and $t = 2$, but this case has been resolved for all k in [2].)

Theorem 1.4. *Let r, t be integers satisfying $r > t \geq 2$ and $(r, t) \neq (3, 2)$. Then the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 7(r-t)+t, 7)$ exists.*

More about the case of k being odd can be found in the concluding remarks.

Remark 1.5. *Shortly after this paper appeared on arXiv, Glock, Kim, Lichev, Pikhurko and Sun [5] determined the value of $\pi(r, t, k)$ for $t = 2$, $k \in \{5, 6, 7\}$ and all $r \geq 3$: they showed that for $k \in \{5, 7\}$, it holds that $\pi(r, 2, k) = 1/(r^2 - r - 1)$ (observe this is the same value as for $k = 3$), while for $k = 6$ it holds that $\pi(3, 2, 6) = 61/330$ and $\pi(r, 2, 6) = 1/(r^2 - r)$ for $r \geq 4$ (observe this extends Theorem 1.2 when $t = 2$ and $k = 6$).*

Organisation. Section 2 introduces the relevant notation and collects some preliminary results, including our key proposition (Proposition 2.4) needed for the proofs of Theorems 1.3 and 1.4, which are proved in Section 4 and Section 5, respectively. Section 3 provides the proof of Theorem 1.2 and finally Section 6 contains some concluding remarks.

Notation. Given an r-graph $\mathcal{F}$, we often think of $\mathcal{F}$ as the edge set $E(\mathcal{F})$. In particular, by $|\mathcal{F}|$ we mean the number of edges in $\mathcal{F}$, and by $e \in \mathcal{F}$ we mean that e is an edge in $\mathcal{F}$. Since the values of r and t will always be clear from the context, we introduce the following terminology. A *k-configuration* denotes a $(k(r-t)+t, k)$-configuration, while a *k^--configuration* denotes a $(k(r-t)+t-1, k)$-configuration. Moreover, we say that a hypergraph is *k-free* (resp. *k^--free*) if it does not contain any k-configuration (resp. k^--configuration).

2 Preliminaries

2.1 Lower bounds

In order to build k-free r-graphs with many edges, the strategy of Glock, Joos, Kim, Kühn, Lichev and Pikhurko [4] consisted of packing many copies of a carefully chosen k-free r-graph of constant size, while making sure not to create any k-configurations using edges from different copies. Before stating their main technical result, we introduce some definitions.

Recall that the *t-shadow* of a hypergraph $\mathcal{F}$, denoted $\partial_t\mathcal{F}$, is the t-graph on $V(\mathcal{F})$ whose edges are the t-subsets of edges in $\mathcal{F}$.

Definition 2.1. Given an r-graph $\mathcal{F}$ and a t-graph J, we say that J is a *supporting t-graph* of $\mathcal{F}$ if $V(J) = V(\mathcal{F})$ and J contains the t-shadow of $\mathcal{F}$. For such $\mathcal{F}$ and J, we define the *non-edge girth* of $(\mathcal{F}, J)$ to be the smallest $g \geq 1$ for which there exists a g-configuration in $\mathcal{F}$ whose vertex set contains a non-edge of J. Equivalently, it is the largest $g \geq 1$ such that for every ℓ-configuration S in $\mathcal{F}$ with $\ell < g$, all t-subsets of $V(S)$ are edges of J. If no such g exists, we set the non-edge girth of $(\mathcal{F}, J)$ to be infinity.

Here is the main technical result in [4].

Theorem 2.2 (Theorem 3.1 in [4])**.** *Fix $k \geq 2$, $r \geq 3$ and $t \in [2, r-1]$. Let $\mathcal{F}$ be an r-graph which is k-free and ℓ^--free for all $\ell \in [2, k-1]$. Let J be a supporting t-graph of $\mathcal{F}$ such that the non-edge girth of $(\mathcal{F}, J)$ is greater than $k/2$. Then,*

$$\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{|\mathcal{F}|}{t!\,|J|}\,.$$

In particular, by choosing $\mathcal{F}$ to be a single r-uniform edge and $J = \binom{V(\mathcal{F})}{t}$, the hypotheses of Theorem 2.2 hold and we get the following corollary.

Corollary 2.3 (Corollary 3.2 in [4])**.** *Fix $k \geq 2$, $r \geq 3$ and $t \in [2, r-1]$. Then,*

$$\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{1}{t!\binom{r}{t}}\,.$$

2.2 Density argument

The approach of Delcourt and Postle [2], while proving Conjecture 1.1 for $r=3$, $t=2$ and any $k \geq 2$, relies on the following reduction: they show that in any sufficiently dense k-free 3-graph, it is possible to find a subgraph with almost the same density which is additionally ℓ^--free for every $\ell \in [2, k-1]$, i.e. having ℓ^--configurations is 'inefficient' for the extremal k-free graph. Here we provide another density-type argument, which we will use in the proof of Theorems 1.3 and 1.4. Before stating the result, we introduce some notation. Given an r-graph $\mathcal{F}$, define $J(\mathcal{F})$ to be the t-graph with $V(\mathcal{F})$ as vertex set and where a t-subset $T \subseteq V(\mathcal{F})$ is an edge of $J(\mathcal{F})$ if and only if there exists an ℓ-configuration for some $\ell \in [\lfloor k/2 \rfloor]$ whose vertex set contains T. Observe that, since every edge is a 1-configuration, $J(\mathcal{F})$ contains the t-shadow of $\mathcal{F}$. Therefore $J(\mathcal{F})$ is a supporting t-graph of $\mathcal{F}$ (recall Definition 2.1). Moreover, its non-edge girth is greater than $\lfloor k/2 \rfloor$.

Proposition 2.4. *Suppose that for every $\varepsilon > 0$ and large enough n, for every k-free n-vertex r-graph $\mathcal{F}$ with $|\mathcal{F}| \geq \left(\binom{r}{t}^{-1} + \varepsilon\right)\binom{n}{t}$ there exist subhypergraphs $\mathcal{F}_2 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}$ such that $|\mathcal{F}_1| \geq |\mathcal{F}| - O(n^{t-1})$, $\mathcal{F}_2$ is ℓ^--free for every $\ell \in [2, k-1]$, and*

$$\frac{|\mathcal{F}_2|}{|J(\mathcal{F}_2)|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|}\,. \tag{1}$$

Then the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$ exists.

Proof. Define α to satisfy $\frac{\alpha}{t!} = \limsup_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$ and observe that, since Corollary 2.3 gives that $\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{1}{t!}\binom{r}{t}^{-1}$, it holds that $\alpha \geq \binom{r}{t}^{-1}$. If we have equality, we are done. Therefore, we can assume the inequality is strict and thus for small enough $\varepsilon > 0$ we have $\alpha > \binom{r}{t}^{-1} + \varepsilon$. Given the definition of α, for every $n \in \mathbb{N}$, there exist $m \geq n$ and an m-vertex k-free r-graph $\mathcal{F}$ with $|\mathcal{F}| \geq (\alpha - \varepsilon)\binom{m}{t}$. Owing to the assumptions of the proposition, there exist subhypergraphs $\mathcal{F}_2 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}$ such that $|\mathcal{F}_1| \geq |\mathcal{F}| - O(m^{t-1})$, $\mathcal{F}_2$ is ℓ^--free for every $\ell \in [2, k-1]$, and $\frac{|\mathcal{F}_2|}{|J(\mathcal{F}_2)|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|}$. As observed above, $J(\mathcal{F}_2)$ is a supporting t-graph of $\mathcal{F}_2$ and its non-edge girth is greater than $k/2$. Therefore, by Theorem 2.2, we have

$$\begin{aligned}\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) &\geq \frac{|\mathcal{F}_2|}{t! \cdot |J(\mathcal{F}_2)|} \geq \frac{|\mathcal{F}_1|}{t! \cdot |J(\mathcal{F}_1)|} \\ &\geq \frac{(\alpha-\varepsilon)\binom{m}{t} - O(m^{t-1})}{t! \cdot \binom{m}{t}} = \frac{\alpha - \varepsilon}{t!} - O(m^{-1})\,,\end{aligned}$$

using that $|J(\mathcal{F}_1)| \leq \binom{m}{t}$ for the last inequality. Since ε can be made arbitrarily small and m arbitrarily large, the conclusion easily follows from

$$\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{\alpha}{t!} = \limsup_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)\,. \qquad \square$$

Remark 2.5. *We remark that if $\mathcal{F}_1$ is an r-graph with $|\mathcal{F}_1| \geq \binom{r}{t}^{-1}|J(\mathcal{F}_1)|$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$, then the condition*

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_2)| \geq \binom{r}{t}(|\mathcal{F}_1| - |\mathcal{F}_2|) \tag{2}$$

implies Condition (1). Indeed, writing $x_1 = |J(\mathcal{F}_1)|$, $y_1 = |\mathcal{F}_1|$, $x_2 = |J(\mathcal{F}_2)|$, $y_2 = |\mathcal{F}_2|$ and $\alpha = \binom{r}{t}$, we have $x_1 \leq \alpha y_1$, $x_2 \leq x_1$ and $y_2 \leq y_1$ by assumption. Moreover, by (2), $x_2 \leq x_1 - \alpha(y_1 - y_2) \leq \alpha y_2$. Therefore, using (2) again, which is equivalent to $\alpha y_1 - x_1 \leq \alpha y_2 - x_2$, together with $x_2 \leq \alpha y_2$ and $x_2 \leq x_1$, we have

$$\begin{aligned}\alpha(x_2 y_1 - x_1 y_2) &= x_2(\alpha y_1 - x_1) + x_1 x_2 - \alpha x_1 y_2 \\ &\leq x_2(\alpha y_2 - x_2) + x_1(x_2 - \alpha y_2) = (x_1 - x_2)(x_2 - \alpha y_2) \leq 0.\end{aligned}$$

This implies $x_1 y_2 \geq x_2 y_1$, which in turn is equivalent to (1).

2.3 A useful lemma

In order to apply the density argument of Proposition 2.4, for any given k-free n-vertex r-graph $\mathcal{F}$, we need to find a subhypergraph which is ℓ^--free for each $\ell \in [2, k-1]$ and satisfies some additional properties. It turns out that, for some values of ℓ, there is a simple argument which shows that $\mathcal{F}$ can be made ℓ^--free by removing only $O(n^{t-1})$ edges. This is established, together with additional properties, by the following lemma.

Lemma 2.6. *Let r, k and t be fixed positive integers. Let $\mathcal{F}$ be a k-free n-vertex r-graph. Then there exists a subhypergraph $\mathcal{F}'$ of $\mathcal{F}$ such that the following holds.*

(P1) *$\mathcal{F}'$ is ℓ^--free for every $\ell \in [2, k]$ with $\ell|(k-1)$ or $\ell|k$;*

(P2) *there is no 3^--configuration in $\mathcal{F}'$ which contains a 2-configuration;*

(P3) *for every positive integers a and b with $a + b = k$, every a^--configuration and every b-configuration of $\mathcal{F}'$ are edge-disjoint;*

(P4) *$|\mathcal{F}'| \geq |\mathcal{F}| - O(n^{t-1})$.*

Proof. We show that we can get a subhypergraph of $\mathcal{F}$ which satisfies (P1), (P2) and (P3) by removing $O(n^{t-1})$ edges, which in turn will imply (P4) as well.

Observe that, since $\mathcal{F}$ is k-free, $\mathcal{F}$ is also k^--free. Moreover we show that $\mathcal{F}$ can be made $(k-1)^-$-free by removing $O(n^{t-1})$ edges. Let $\mathcal{S}$ be a maximal collection of pairwise edge-disjoint $(k-1)^-$-configurations of $\mathcal{F}$. If $|\mathcal{S}| > \binom{n}{t-1}$, then there exists a set $T \subseteq V(\mathcal{F})$ of size $t-1$ which is contained in the $(t-1)$-shadow of two $(k-1)^-$-configurations S_1 and S_2 in $\mathcal{S}$. Let $e \in S_2$ be an edge such that $T \subseteq e$. Then $S_1 \cup \{e\}$ is a k-configuration in $\mathcal{F}$, being a collection of k edges spanning at most $[(k-1)(r-t)+t-1]+(r-|T|) = k(r-t)+t$ vertices, a contradiction to $\mathcal{F}$ being k-free. Therefore $|\mathcal{S}| \leq \binom{n}{t-1}$ and, by removing from $\mathcal{F}$ all edges of each $S \in \mathcal{S}$, we obtain a subhypergraph $\mathcal{F}_0 \subseteq \mathcal{F}$ which is $(k-1)^-$-free and satisfies $|\mathcal{F}_0| \geq |\mathcal{F}| - O(n^{t-1})$.

Let $2 \leq \ell < k-1$ with $\ell|(k-1)$ (resp. $\ell|k$). Let $j > 1$ be the positive integer such that $\ell \cdot j = k-1$ (resp. $\ell \cdot j = k$). Let $\mathcal{S}_\ell$ be a maximal collection of pairwise edge-disjoint ℓ^--configurations in $\mathcal{F}_0$. If $|\mathcal{S}_\ell| > (j-1) \cdot \binom{n}{t-1}$, then there exists a set $T \subseteq V(\mathcal{F})$ of size $t-1$ which is contained in the vertex set of j distinct ℓ^--configurations $S_1, \ldots, S_j$ in $\mathcal{S}_\ell$. Then $S_1 \cup \cdots \cup S_j$ is a $(k-1)^-$-configuration of $\mathcal{F}_0$, being a collection of $\ell \cdot j = k-1$ edges spanning at most $j[\ell(r-t)+t-1] - (j-1)|T| = j\ell(r-t)+t-1 = (k-1)(r-t)+t-1$ vertices (resp. a k^--configuration). This is a contradiction to $\mathcal{F}_0$ being $(k-1)^-$-free (resp. k^--free). Therefore $|\mathcal{S}_\ell| = O(n^{t-1})$ for all relevant ℓ.

If $k \equiv 0$ or $1 \pmod 3$, then (P1) would trivially imply (P2), as $\mathcal{F}'$ would not contain any 3^--configurations, and we set $\mathcal{S}' := \emptyset$. If that is not the case, namely if $k \equiv 2 \pmod 3$, define $\mathcal{S}'$ to be a maximal collection of pairwise edge-disjoint 3^--configurations containing a 2-configuration. We claim that $|\mathcal{S}'| \leq \frac{k-2}{3} \cdot \binom{n}{t-1}$. Indeed, otherwise, there is a $(t-1)$-subset $T \subseteq V(\mathcal{F})$ and $(k-2)/3+1 = (k+1)/3$ many 3^--configurations $S_1, \ldots, S_{(k+1)/3} \in \mathcal{S}'$, where S_i contains a 2-configuration S'_i satisfying $T \subseteq V(S'_i)$. Then $S_1 \cup \ldots \cup S_{(k-2)/3} \cup S'_{(k+1)/3}$ is a k-configuration, being a collection of k edges spanning at most $((k-2)/3) \cdot [3(r-t)+t-1] + [2(r-t)+t] - ((k-2)/3) \cdot (t-1) = k(r-t)+t$, a contradiction to $\mathcal{F}_0$ being k-free. Therefore $|\mathcal{S}'| = O(n^{t-1})$.

Let a and b be positive integers with $a + b = k$. Let H_a be the collection of edges contained in a^--configurations of $\mathcal{F}_0$ and let $\mathcal{S}_{a,b}$ be a maximal collection of pairwise edge-disjoint b-configurations of $\mathcal{F}_0$ containing an edge of H_a. If $|\mathcal{S}_{a,b}| > a \cdot \binom{n}{t-1}$, then there exists a set $T \subseteq V(\mathcal{F})$ of size $t-1$ and $a+1$ distinct b-configurations $S_1, \ldots, S_{a+1}$ in $\mathcal{S}_{a,b}$ such that there exists $e_i \in S_i \cap H_a$ with $T \subseteq e_i$ for every $i \in [a+1]$. By definition of H_a, there exists $f_2, \ldots, f_a \in \mathcal{F}_0$ such that $S' := \{e_1, f_2, \ldots, f_a\}$ is an a^--configuration and, without loss of generality, we assume that S_{a+1} and S' are edge-disjoint. Then $S_{a+1} \cup S'$ is a k-configuration of $\mathcal{F}_2$, being a collection of $b+a=k$ edges spanning at most $[b(r-t)+t]+[a(r-t)+t-1]-|T|=k(r-t)+t$, a contradiction to $\mathcal{F}_0$ being k-free. Therefore $|\mathcal{S}_{a,b}|=O(n^{t-1})$.

Let $\mathcal{F}'$ be the subhypergraph of $\mathcal{F}_0$ which is obtained by removing all the edges of each $S\in\mathcal{S}_\ell$ for every $2\le \ell<k-1$ with $\ell|(k-1)$ or $\ell|k$, all the edges of each $S\in\mathcal{S}'$, and all the edges of each $S\in\mathcal{S}_{a,b}$ for every positive integers a and b with $a+b=k$. Then $\mathcal{F}'$ satisfies (P1), (P2), (P3) and (P4). $\square$

We remark that Proposition 2.4 and Lemma 2.6 offer short proofs that Conjecture 1.1 holds for $k=2$ and $k=3$. Indeed, the case $k=2$ is immediate. For $k=3$, given a 3-free r-graph $\mathcal{F}$, Lemma 2.6 gives a subhypergraph $\mathcal{F}'\subseteq\mathcal{F}$ which is 2^--free and satisfies $|\mathcal{F}'|\ge|\mathcal{F}|-O(n^{t-1})$. We can then apply Proposition 2.4 with $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}'$.

3 Proof of Theorem 1.2 (Conjecture 1.1 for k even)

In this section, we prove Theorem 1.2, which asserts that $\lim_{n\to\infty} n^{-t}f^{(r)}(n;k(r-t)+t,k)=\frac{1}{t!}\binom{r}{t}^{-1}$, for k even and r sufficiently large in terms of t and k. We do that by showing that $\liminf_{n\to\infty} n^{-t}f^{(r)}(n;k(r-t)+t,k)\ge\frac{1}{t!\binom{r}{t}}$, which follows directly from Corollary 2.3, and that $\limsup_{n\to\infty} n^{-t}f^{(r)}(n;k(r-t)+t,k)\le\frac{1}{t!\binom{r}{t}}$, which follows from Lemma 2.6 and the inequality and claims provided below.

Claim 3.1. *Suppose that r,t,k are integers satisfying $t,k\ge 2$ and $r\ge t+(k^3\cdot t!)^{1/t}$. Then*

$$\binom{2r-t}{t}-\left[2\binom{r}{t}-1\right]-(k-3)\ge(k-2)^3.$$

Proof. Notice that

$$\begin{aligned}\binom{2r-t}{t}&=\frac{1}{t!}\cdot\prod_{i=0}^{t-1}(2r-t-i)=\frac{1}{t!}\cdot(2r-2t+1)(2r-2t+2)\cdot\prod_{i=0}^{t-3}(r-i+r-t)\\&\ge\frac{1}{t!}\cdot 2\cdot(r-t+1)(r-t+2)\cdot\left[\prod_{i=0}^{t-3}(r-i)+\prod_{i=0}^{t-3}(r-t)\right]\\&\ge 2\binom{r}{t}+\frac{2}{t!}(r-t)^t\,,\end{aligned}$$

where in the second line we used that $2\le t\le r-1$. This in turn gives

$$\binom{2r-t}{t}-2\binom{r}{t}\ge\frac{2}{t!}\cdot(r-t)^t\ge k^3\ge(k-2)^3+(k-3)\,,$$

where the second inequality follows from $r\ge t+(k^3\cdot t!)^{1/t}$. $\square$

Claim 3.2. *Let $\mathcal{F}$ be a k-free r-graph and $e\in\mathcal{F}$. Then the number of 2-configurations of $\mathcal{F}$ containing e is at most $k-2$.*

Proof. If there were $k-1$ distinct 2-configurations $\{e,e_i\}$ for $i\in[k-1]$, then $S:=\{e,e_1,\dots,e_{k-1}\}$ would be a k-configuration of $\mathcal{F}$, being a collection of k edges spanning at most $r+(k-1)(r-t)=k(r-t)+t$ vertices, a contradiction to $\mathcal{F}$ being k-free. $\square$

Claim 3.3. *Let k be an even integer, $\mathcal{F}$ a k-free r-graph and $T\subseteq V(\mathcal{F})$ with $|T|=t$. Then the number of 2-configurations whose vertex set contains T is at most $(k-2)^2$.*

Proof. Let $\mathcal{S}$ be a maximal collection of pairwise edge-disjoint 2-configurations of $\mathcal{F}$ whose vertex set contains T. We have $|\mathcal{S}| \le (k-2)/2$, as otherwise there would exist distinct 2-configurations $S_1, \dots, S_{k/2}$ in $\mathcal{S}$ and $S_1 \cup \cdots \cup S_{k/2}$ would give a k-configuration of $\mathcal{F}$, being a collection of k edges spanning at most $(2r-t)k/2 - (k/2-1)|T| = k(r-t)+t$ vertices. By maximality of $\mathcal{S}$, any 2-configuration of $\mathcal{F}$ whose vertex set contains T, must contain an edge which belongs to some $S \in \mathcal{S}$. There are $2|\mathcal{S}| \le k-2$ such edges and, for each of them, by Claim 3.2, the number of 2-configurations of $\mathcal{F}$ containing this edge is at most $k-2$. Therefore the number of 2-configurations of $\mathcal{F}$ whose vertex set contains T is at most $(k-2)^2$. □

Given a hypergraph $\mathcal{F}$, we say that a t-set T of $V(\mathcal{F})$ is *covered* exactly i times, if T is contained in exactly i edges of $\mathcal{F}$. We denote by $J_i(\mathcal{F})$ the set of t-subsets of $V(\mathcal{F})$ covered exactly i times, and by $J_{\ge i}(\mathcal{F})$ the set of t-subsets of $V(\mathcal{F})$ covered at least i times.

Claim 3.4. *Let k be a positive even integer and $\mathcal{F}$ be an r-graph which is k-free, 2^--free, and has no 3^--configurations containing a 2-configuration. Write $J_0 := J_0(\mathcal{F})$ and $J_{\ge 2} := J_{\ge 2}(\mathcal{F})$. Then*

$$(k-2)^2|J_0| \ge \left\{ \binom{2r-t}{t} - \left[2\binom{r}{t} - 1 \right] - (k-3) \right\} \cdot |J_{\ge 2}| \, .$$

Proof. We use a double counting argument on the set

$$\mathcal{Q} := \left\{ (S,T) : \begin{array}{c} S \text{ is a 2-configuration of } \mathcal{F}, \\ T \subseteq V(S),\ |T| = t \text{ and } T \in J_0 \end{array} \right\} .$$

Fix $T \subseteq V(\mathcal{F})$ with $|T| = t$. By Claim 3.3, the number of 2-configurations of $\mathcal{F}$ whose vertex set contains T is at most $(k-2)^2$. We conclude that

$$|\mathcal{Q}| \le (k-2)^2 |J_0| \, . \tag{3}$$

Now fix a 2-configuration $S := \{f_1, f_2\}$ and observe that, since $\mathcal{F}$ is 2^--free, S spans precisely $2r - t$ vertices, so f_1 and f_2 share precisely t vertices. We now estimate the number of t-sets $T \subseteq V(S)$ with $T \in J_0$. Since $T \subseteq V(S)$, either T is fully contained in f_1 or in f_2, or intersects both $f_1 \setminus f_2$ and $f_2 \setminus f_1$.

If T is fully contained in f_1, then it does not belong to J_0, as it is covered at least once (by the edge f_1). Clearly, the same argument applies to any T which is fully contained in f_2. Moreover, the number of such t-sets is $2\binom{r}{t} - 1$.

Now we consider those T intersecting both $f_1 \setminus f_2$ and $f_2 \setminus f_1$. If $T \notin J_0$, then there exists $e \in \mathcal{F}$ with $T \subseteq e$, and clearly $e \ne f_1, f_2$. Notice that, $|e \cap V(S)| \le t$, by the assumption that there are no 3^--configurations containing a 2-configuration. As $T \subseteq e \cap V(S)$, we have that $e \cap V(S) = T$. It follows that among the t-sets of $V(S)$ intersecting both $f_1 \setminus f_2$ and $f_2 \setminus f_1$, all but at most $k-3$ belong to J_0. Indeed, otherwise, there would exist pairwise distinct t-sets $T_1, \dots, T_{k-2} \subseteq V(S)$ and pairwise distinct edges $e_1, \dots, e_{k-2}$ with $T_i \subseteq e_i$ for each $i \in [k-2]$. However, $\{e_1, \dots, e_{k-2}, f_1, f_2\}$ would be a k-configuration, being a collection of k edges spanning at most $(k-2)(r-t) + 2r - t = k(r-t) + t$ vertices.

Therefore, for a given 2-configuration S, the number of t-sets $T \subseteq V(S)$ with $T \in J_0$ is at least

$$\binom{2r-t}{t} - \left[2\binom{r}{t} - 1 \right] - (k-3) \, , \tag{4}$$

where the first term stands for the number of t-sets of $V(S)$, while the rest accounts for the arguments above.

Finally, observe that every $T \in J_{\ge 2}$ gives rise to a 2-configuration $\{f_1, f_2\}$ with $T = f_1 \cap f_2$ (we have $T \subseteq f_1 \cap f_2$ by definition, with equality because $\mathcal{F}$ is 2^--free), and these 2-configurations are distinct for different sets T. This shows that the number of 2-configurations of $\mathcal{F}$ is at least $|J_{\geq 2}|$. Using (4), we conclude that

$$|\mathcal{Q}| \geq \left\{ \binom{2r-t}{t} - \left[2\binom{r}{t} - 1\right] - (k-3) \right\} \cdot |J_{\geq 2}| \,. \tag{5}$$

The claim follows from (3) and (5). $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let k be an even integer, $t \geq 2$ an integer and let r be an integer satisfying $r \geq t + (k^3 \cdot t!)^{1/t}$. From Corollary 2.3 we get

$$\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{1}{t!\binom{r}{t}} \,. \tag{6}$$

Let $\mathcal{F}$ be a k-free n-vertex r-graph. By Lemma 2.6, there exists a subhypergraph $\mathcal{F}'$ of $\mathcal{F}$ which is 2^--free, has no 3^--configurations containing a 2-configuration and satisfies $|\mathcal{F}'| \geq |\mathcal{F}| - O(n^{t-1})$. Set $J_i := J_i(\mathcal{F}')$ and $J_{\geq i} := J_{\geq i}(\mathcal{F}')$ and observe that applications of Claim 3.4 and Claim 3.1 give

$$(k-2)^2|J_0| \geq \left\{ \binom{2r-t}{t} - \left[2\binom{r}{t} - 1\right] - (k-3) \right\} \cdot |J_{\geq 2}| \geq (k-2)^3|J_{\geq 2}| \,.$$

Therefore, $|J_0| \geq (k-2) \cdot |J_{\geq 2}|$. Now consider the following set

$$\mathcal{P} := \{(e,T) : e \in \mathcal{F}', T \subseteq e \text{ with } |T| = t\} \,.$$

Then $|\mathcal{P}| = |\mathcal{F}'| \cdot \binom{r}{t}$ and

$$\begin{aligned} |\mathcal{P}| = \sum_{i\geq 1} i \cdot |J_i| &\leq |J_1| + (k-1) \cdot |J_{\geq 2}| \\ &= |J_{\geq 0}| + (k-2) \cdot |J_{\geq 2}| - |J_0| \leq \binom{n}{t}, \end{aligned}$$

where in the first inequality we used that a t-set covered at least k times gives a k-configuration and thus, since $\mathcal{F}'$ is k-free, we have $J_{\geq k} = \emptyset$, while in the last inequality we used $|J_{\geq 0}| = \binom{n}{t}$ and $|J_0| \geq (k-2) \cdot |J_{\geq 2}|$. We conclude that

$$|\mathcal{F}| \leq |\mathcal{F}'| + O(n^{t-1}) = |\mathcal{P}| \cdot \binom{r}{t}^{-1} + O(n^{t-1}) \leq \binom{n}{t} \cdot \binom{r}{t}^{-1} + O(n^{t-1}) \,,$$

which allows us to establish that

$$\limsup_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \leq \frac{1}{t!\binom{r}{t}} \,. \tag{7}$$

The theorem follows from (6) and (7). $\square$

4 Proof of Theorem 1.3 (Conjecture 1.1 for $k = 5$)

In this section we prove Theorem 1.3, asserting that the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 5(r-t)+5)$ exists, for $2 \leq t < r$. Our proof uses our density argument (Proposition 2.4). We recall that $J(\mathcal{F})$ is the t-graph on $V(\mathcal{F})$ whose edges are t-subsets of ℓ-configurations in $\mathcal{F}$ with $\ell \leq \lfloor k/2 \rfloor$ (this is defined above Proposition 2.4).

Proof of Theorem 1.3. Let $\varepsilon > 0$ and $\mathcal{F}$ be a 5-free n-vertex r-graph with $|\mathcal{F}| \geq \left(\binom{r}{t}^{-1} + \varepsilon\right)\binom{n}{t}$, and suppose that n is large. By Lemma 2.6, there is a subhypergraph $\mathcal{F}_1 \subseteq \mathcal{F}$ which is 2^--free, 4^--free and 5-free, satisfies

$$|\mathcal{F}_1| \geq |\mathcal{F}| - O(n^{t-1}) \geq \binom{r}{t}^{-1}\binom{n}{t} \geq \binom{r}{t}^{-1}|J(\mathcal{F}_1)|, \tag{8}$$

and where any 2-configuration and any 3^--configuration are edge-disjoint.

Claim 4.1. *Let $\mathcal{G} \subseteq \mathcal{F}_1$ and suppose that S is a 3^--configuration in $\mathcal{G}$. Then the following holds with $\mathcal{G}' := \mathcal{G} \setminus S$.*

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right) .$$

Proof. Observe that, by definition of $J(S)$, we have that $T \in J(S)$ if and only there exists an edge $e \in S$ with $T \subseteq e$ or there exists a 2-configuration S' in S whose vertex set contains T. Since in $\mathcal{G}$ any 2-configuration and any 3^--configuration are edge-disjoint, we can rule out the second option. Moreover, a set of size t cannot be in more than one edge of S as, otherwise, S would contain a 2-configuration, which cannot happen for the same reason. Since S has three edges, we conclude that $|J(S)| = 3\binom{r}{t}$.

Let $T \in J(S)$. Then clearly $T \in J(\mathcal{G})$, and we aim to show that $T \notin J(\mathcal{G}')$. For that, note that $T \in J(\mathcal{G}')$ if and only if there exists an edge $e \in \mathcal{G}'$ with $T \subseteq e$ or there exists a 2-configuration S' of $\mathcal{G}'$ whose vertex set contains T. The first option cannot happen as, otherwise, $S \cup \{e\}$ would be a 4^--configuration of $\mathcal{G}$, being a collection of four edges spanning at most $(3r - 2t - 1) + r - t = 4(r - t) + t - 1$ vertices, a contradiction to $\mathcal{F}_1$ being 4^--free. Similarly, we can rule out the second option as, otherwise, using that S and S' are edge-disjoint, $S \cup S'$ would be a 5^--configuration of $\mathcal{G}$, being a collection of five edges spanning at most $(3r - 2t - 1) + (2r - t) - t = 5(r - t) + t - 1$ vertices, a contradiction to $\mathcal{F}_1$ being 5^--free.

Therefore $|J(\mathcal{G})| - |J(\mathcal{G}')| \geq |J(S)| = 3\binom{r}{t}$ and the claim follows. □

By applying Claim 4.1 repeatedly, we can find a subhypergraph $\mathcal{F}_2 \subseteq \mathcal{F}_1$, which is 3^--free and satisfies

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_2)| \geq \binom{r}{t}\left(|\mathcal{F}_1| - |\mathcal{F}_2|\right) .$$

By Remark 2.5 and (8), it follows that

$$\frac{|\mathcal{F}_2|}{|J(\mathcal{F}_2)|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|} .$$

Notice that $\mathcal{F}_2$ is ℓ^--free for $\ell \in \{2, 3, 4\}$ and 5-free. Thus Proposition 2.4 implies that the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 5(r - t) + t, 5)$ exists. □

5 Proof of Theorem 1.4 (Conjecture 1.1 for $k = 7$)

This section is concerned with the proof of Theorem 1.4, which asserts that the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 7(r-t) + t, 7)$ exists for $r > t \geq 2$ and $(r, t) \neq (3, 2)$. We use our density argument (Proposition 2.4), together with the following two inequalities.

Claim 5.1. *Let r, t be integers such that $3 \leq t < r$ or $t = 2$ and $r \geq 4$. Then*

$$\binom{3r - 2t}{t} - 4 \geq 3\binom{r}{t} .$$

Proof. The claimed inequality can be checked directly for $t = 2$, so suppose that $t \geq 3$.

$$\begin{aligned}\binom{3r-2t}{t} &= \frac{1}{t!} \cdot \prod_{i=0}^{t-1} (3r - 2t - i) \\ &= \frac{1}{t!} \cdot (3r - 3t + 1)(3r - 3t + 2)(3r - 3t + 3) \cdot \prod_{i=0}^{t-4} (r - i + 2r - 2t) \\ &\geq \frac{1}{t!} \cdot 4(r - t + 1)(r - t + 2)(r - t + 3) \cdot \prod_{i=0}^{t-4} (r - i) \\ &= 4\binom{r}{t} \geq 3\binom{r}{t} + 4 \,.\end{aligned}$$

Here in the first inequality we used the inequality $(3x + 1)(3x + 2)(3x + 3) \geq 4(x + 1)(x + 2)(x + 3)$ for $x \geq 1$, which can be checked directly. In the last inequality we used that $\binom{r}{t} \geq \binom{t+1}{t} = t + 1 \geq 4$. □

Claim 5.2. *Let r, t be integers such that $3 \leq t < r$ or $t = 2$ and $r \geq 4$. Then*

$$\binom{2r-t}{t} \geq 2\binom{r}{t} + 2.$$

Proof. Let A, B, C be pairwise disjoint sets of sizes $r - t, r - t, t$, respectively. Then $\binom{2r-t}{t}$ is the number of t-subsets of $A \cup B \cup C$. This is at least the number of t-subsets of either $A \cup C$ or $B \cup C$, of which there are $2\binom{r}{t} - 1$, plus the number of t-subsets of $A \cup B \cup C$ consisting of one vertex from each of A and B and $t-2$ vertices from C, of which there are $(r - t)^2 \binom{t}{t-2} \geq 3$, using that either $t \geq 3$ or $r - t \geq 2$. Altogether, we have that $\binom{2r-t}{t} \geq 2\binom{r}{t} - 1 + 3 = 2\binom{r}{t} + 2$, as required. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\varepsilon > 0$ and $\mathcal{F}$ be a 7-free n-vertex r-graph with $|\mathcal{F}| \geq \left(\binom{r}{t}^{-1} + \varepsilon\right)\binom{n}{t}$, and suppose that n is large.

Apply Lemma 2.6 to get a subhypergraph $\mathcal{F}_1 \subseteq \mathcal{F}$ which is 2^--free, 3^--free, 6^--free and 7-free, satisfies

$$|\mathcal{F}_1| \geq |\mathcal{F}| - O(n^{t-1}) \geq \binom{r}{t}^{-1}\binom{n}{t} \geq \binom{r}{t}^{-1} |J(\mathcal{F}_1)|, \tag{9}$$

and where any 2-configuration and any 5^--configuration are edge-disjoint, and any 3-configuration and any 4^--configuration are edge-disjoint. Now we prove some structural claims on subhypergraphs of $\mathcal{F}_1$.

Claim 5.3. *Let $\mathcal{G} \subseteq \mathcal{F}_1$ and suppose that S is a 3-configuration in $\mathcal{G}$ contained in a 4-configuration in $\mathcal{G}$. Then the following holds with $\mathcal{G}' := \mathcal{G} \setminus S$.*

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{r}{t} \left(|\mathcal{G}| - |\mathcal{G}'|\right) .$$

Proof. Write $S := \{e_1, e_2, e_3\}$ and let $e_4 \in \mathcal{G}$ be such that $\{e_1, e_2, e_3, e_4\}$ is a 4-configuration. Let $T' := V(S) \cap e_4$ and observe that $|T'| = t$. Indeed, $|T'| \geq t$ follows from the fact that S is a 3-configuration but not a 3^--configuration, implying that $|V(S)| = 3r - 2t$, and $S \cup \{e_4\}$ is a 4-configuration, while $|T'| \leq t$ follows from the fact that otherwise $S \cup \{e_4\}$ would be a 4^--configuration, a contradiction to any 3-configuration and any 4^--configuration of $\mathcal{F}_1$ being edge-disjoint.

We now lower bound $|J(\mathcal{G})| - |J(\mathcal{G}')|$. Let $T \subseteq V(S)$ satisfy $|T| = t$. Since S is a 3-configuration, we have $T \in J(\mathcal{G})$. For T to be in $J(\mathcal{G}')$ there must be an ℓ-configuration in $\mathcal{G}'$ with $\ell \in [3]$ whose vertex set contains T. We now prove the following assertions, to help us bound the number of times each of these options can happen.

(i) excluding T', no t-subset of $V(S)$ is contained in the vertex set of a 3-configuration of $\mathcal{G}'$;

(ii) at most one t-subset of $V(S)$ is contained in the vertex set of a 2-configuration of $\mathcal{G}'$;

(iii) excluding T', at most two t-subsets of $V(S)$ are contained in an edge of $\mathcal{G}'$.

We show (i) as follows. Any 3-configuration S' in $\mathcal{G}'$ satisfies $|V(S) \cap V(S')| \leq t$ as otherwise $S \cup S'$ would be a 6^--configuration. Therefore if a t-subset $T \neq T'$ of $V(S)$ is contained in a 3-configuration S' in $\mathcal{G}'$, then $e_4 \notin S'$ and $S \cup S' \cup \{e_4\}$ is a 7-configuration of $\mathcal{G}$, a contradiction.

For (ii) we argue as follows. First observe that any 2-configuration S' in $\mathcal{G}'$ satisfies $|V(S) \cap V(S')| \leq t$ as otherwise $S \cup S'$ would be a 5^--configuration of $\mathcal{G}$, which is a contradiction as any 5^--configuration and any 2-configuration of $\mathcal{G}$ are edge-disjoint. Now suppose there are two distinct t-subsets T_1 and T_2 of $V(S)$ and two 2-configurations S_1 and S_2 in $\mathcal{G}'$ with $T_i \subseteq V(S_i)$ for $i \in [2]$. Observe that $S_1 \neq S_2$ as any 2-configuration in $\mathcal{G}'$ intersects $V(S)$ in no more than t vertices, as argued above. If S_1 and S_2 were edge-disjoint, then $S_1 \cup S_2 \cup S$ would be a 7-configuration of $\mathcal{G}$, a contradiction. If S_1 and S_2 were not edge-disjoint, then $S_1 \cup S_2$ would be a 3-configuration intersecting $V(S)$ in more than t vertices, but then $S_1 \cup S_2 \cup S$ would be a 6^--configuration of $\mathcal{G}$, a contradiction.

Finally, we prove (iii). Any edge not in S intersects $V(S)$ in at most t vertices, as 3-configurations and 4^--configurations of $\mathcal{G}$ are edge-disjoint. Suppose there were three distinct t-subsets T_1, T_2 and T_3 of $V(S)$, all distinct from T', and three (distinct) edges f_1, f_2 and f_3 not in S with $T_i \subseteq f_i$ for $i \in [3]$. Then $e_4 \notin \{f_1, f_2, f_3\}$ and $S \cup \{e_4, f_1, f_2, f_3\}$ would be a 7-configuration of $\mathcal{G}$, a contradiction.

Recall that $|V(S)| = 3r - 2t$, so there are $\binom{3r-2t}{t}$ subsets of $V(S)$ of size t. Taking (i), (ii) and (iii) into account, and using Claim 5.1, we get

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{3r-2t}{t} - 1 - 3 \geq 3\binom{r}{t} = \binom{r}{t}(|\mathcal{G}| - |\mathcal{G}'|),$$

as claimed. □

By repeatedly applying Claim 5.3, we get a subhypergraph $\mathcal{F}_2 \subseteq \mathcal{F}_1$ satisfying

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_2)| \geq \binom{r}{t}(|\mathcal{F}_1| - |\mathcal{F}_2|), \tag{10}$$

which has no 3-configuration contained in a 4-configuration.

Claim 5.4. *Let $\mathcal{G} \subseteq \mathcal{F}_2$. Suppose that S is a 4^--configuration in $\mathcal{G}$. Then there exists a non-empty subset $S' \subseteq S$, such that the following holds with $\mathcal{G}' := \mathcal{G} \setminus S'$.*

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{r}{t}(|\mathcal{G}| - |\mathcal{G}'|). \tag{11}$$

Proof. We start by observing that, for $e, e' \in \mathcal{G}$, if $e \in S$ and $\{e, e'\}$ is a 2-configuration, then $e' \in S$. Indeed, otherwise, the 5^--configuration $S \cup \{e'\}$ and the 2-configuration $\{e, e'\}$ would not be edge-disjoint, a contradiction. Therefore, either S contains a 2-configuration or the edges of S are not involved in any 2-configuration of $\mathcal{G}$. Since S contains no 3-configurations (by every 4^--configuration being edge-disjoint of all 3-configurations in $\mathcal{G}$), we have the following three cases: S contains no 2-configurations; S contains a single 2-configuration; and S can be partitioned into two 2-configurations. We consider each case separately.

Case 1. S contains no 2-configurations. Let e be an edge of S, and set $\mathcal{S}' := \{e\}$ and $\mathcal{G}' := \mathcal{G} \setminus \{e\}$. We claim that $J(\mathcal{G}) \setminus J(\mathcal{G}')$ contains all t-subsets of $V(S')$, which would prove (11) in this case. Since any such t-subset belongs to $J(\mathcal{G})$, this follows once we show that

(i) no t-subset of e is contained in the vertex set of a 3-configuration of $\mathcal{G}'$;

(ii) no t-subset of e is contained in the vertex set of a 2-configuration of $\mathcal{G}'$;

(iii) no t-subset of e is contained in an edge of $\mathcal{G}'$.

Fact (i) holds since, by assumption, any 3-configuration is edge-disjoint of S and thus, if it shares t vertices with e, its union with S would give a 7^{-}-configuration, a contradiction. For (ii), recall that if there was a t-subset of e contained in a 2-configuration S'' then $e \notin S''$ and e would belong to both the 3-configuration $S'' \cup \{e\}$ and the 4^{-}-configuration S, a contradiction. Finally, (iii) holds as otherwise e would belong to a 2-configuration of $\mathcal{G}$, a contradiction to the assumption that S contains no 2-configurations and its edges are thus not involved in 2-configurations in $\mathcal{G}$.

Case 2. S contains a single 2-configuration S'. Set $\mathcal{G}' := \mathcal{G} \setminus S'$. Let $T \subseteq V(S')$ satisfy $|T| = t$. Since S' is a 2-configuration, we have $T \in J(\mathcal{G})$. For T to be in $J(\mathcal{G}')$, there must be an ℓ-configuration in $\mathcal{G}'$ with $\ell \in [3]$ whose vertex set contains T. We prove the following assertions, to help us bound the number of times this can happen.

(i) no t-subset of $V(S')$ is contained in the vertex set of a 3-configuration of $\mathcal{G}'$;

(ii) no t-subset of $V(S')$ is contained in the vertex set of a 2-configuration of $\mathcal{G}'$;

(iii) no t-subset of $V(S')$ is contained in an edge of $\mathcal{G}'$.

Indeed, (i) can be proved as in the previous case. For (ii), if S'' is a 2-configuration of $\mathcal{G}'$ which intersects $V(S')$ in (at least) t vertices then, by the assumption on S, the configurations S and S'' are edge-disjoint, but then $S \cup S''$ is a 6^{-}-configuration in $\mathcal{G}$, a contradiction. Finally, (iii) holds since $e \in \mathcal{G}'$ intersects $V(S')$ in at most $t-1$ vertices, as otherwise S and $S' \cup \{e\}$ are 4^{-}- and 3-configurations that are not edge-disjoint.

By (i), (ii), (iii) and Claim 5.2, we have

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{2r-t}{t} \geq 2\binom{r}{t} = \binom{r}{t} \cdot \left(|\mathcal{G}| - |\mathcal{G}'|\right).$$

Case 3. S can be partitioned into two 2-configurations S_1, S_2. Set $S' := S$ and $\mathcal{G}' := \mathcal{G} \setminus S'$. Let $\mathcal{J}$ be the collection of t-sets which are subsets of either $V(S_1)$ or $V(S_2)$. Note that if T is in the t-shadow of S_1 then T is not a subset of $V(S_2)$ (otherwise, S would contain a 3-configuration). Thus,

$$|\mathcal{J}| \geq |\partial_t S_1| + \binom{|V(S_2)|}{t} = 2\binom{r}{t} - 1 + \binom{2r-t}{t} \geq 4\binom{r}{t}, \tag{12}$$

using Claim 5.2. Notice that $\mathcal{J} \subseteq J(\mathcal{G})$, as its elements are t-subsets of vertex sets of 2-configurations. As usual, we claim that

(i) no t-set in $\mathcal{J}$ is contained in the vertex set of a 3-configuration of $\mathcal{G}'$;

(ii) no t-set in $\mathcal{J}$ is contained in the vertex set of a 2-configuration of $\mathcal{G}'$;

(iii) no t-set in $\mathcal{J}$ is contained in an edge of $\mathcal{G}'$.

Assertion (i) can be proved as in the first case. For (ii), if S'' is a 2-configuration in $\mathcal{G}'$ whose vertex set contains a t-set in $\mathcal{J}$, then $S \cup S''$ is a 6^--configuration in $\mathcal{G}$, a contradiction. Finally, for (iii), if e is an edge containing a t-set in $\mathcal{J}$ then $S \cup \{e\}$ contains a 3-configuration, a contradiction to the disjointness of 4^-- and 3-configurations.

It follows from (i), (ii), (iii) and (12) that

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq |\mathcal{J}| \geq 4\binom{r}{t} = \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right) . \qquad \square$$

By repeatedly applying Claim 5.4, we get a subhypergraph $\mathcal{F}_3 \subseteq \mathcal{F}_2$ which is 4^--free and satisfies

$$|J(\mathcal{F}_2)| - |J(\mathcal{F}_3)| \geq \binom{r}{t}\left(|\mathcal{F}_2| - |\mathcal{F}_3|\right). \tag{13}$$

Claim 5.5. *Let $\mathcal{G} \subseteq \mathcal{F}_3$ and suppose there exists a 5^--configuration S of $\mathcal{G}$. Then the following holds with $\mathcal{G}' := \mathcal{G} \setminus S$.*

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right) .$$

Proof. Let $\mathcal{J}$ be the t-shadow of S. Then $\mathcal{J} \subseteq J(\mathcal{G})$ and $|\mathcal{J}| = 5\binom{r}{t}$, as a set of size t cannot be in more than one edge of S (otherwise the 5^--configuration S would contain a 2-configuration, a contradiction).

Next we show that if $T \in \mathcal{J}$, then $T \notin J(\mathcal{G}')$; for that it is enough to prove that T is not contained in any ℓ-configuration of $\mathcal{G}'$ with $\ell \in [3]$. For $\ell = 1$, this follows from the fact that any 2-configuration and any 5^--configuration of $\mathcal{G}$ are edge-disjoint. Similarly, for $\ell = 2$ this holds since any 2-configuration is edge-disjoint of S and thus, if it shares t vertices with S, its union with S would give a 7^--configuration, a contradiction. Finally, for $\ell = 3$, we use that, from Claim 5.3, a 3-configuration cannot be contained in any 4-configuration of $\mathcal{G}$. Therefore, we get

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq |\mathcal{J}| = 5\binom{r}{t} = \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right) . \qquad \square$$

By repeatedly applying Claim 5.4, we get a subhypergraph $\mathcal{F}_4 \subseteq \mathcal{F}_3$ which is 5^--free and satisfies

$$|J(\mathcal{F}_3)| - |J(\mathcal{F}_4)| \geq \binom{r}{t}\left(|\mathcal{F}_3| - |\mathcal{F}_4|\right). \tag{14}$$

By summing up (10), (13) and (14), we get

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_4)| \geq \binom{r}{t}\left(|\mathcal{F}_1| - |\mathcal{F}_4|\right).$$

Thus, using Remark 2.5 and (9),

$$\frac{|\mathcal{F}_4|}{|J(\mathcal{F}_4)|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|} .$$

Notice that $\mathcal{F}_4$ is ℓ^--free for $\ell \in \{2,3,4,5,6\}$ and 7-free. Thus, by Proposition 2.4, the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 7(r-t)+t, 7)$ exists. $\square$

6 Conclusion

Recall that we defined $\pi(r,t,k) := \lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$ (if the limit exists). Theorem 1.2 establishes that $\pi(r,t,k) = \frac{1}{t!}\binom{r}{t}^{-1}$ when k is even and $r \geq t + (k^3 \cdot t!)^{1/t}$. It would be interesting to determine, for fixed even k, what is the smallest r such that $\pi(r,t,k) = \frac{1}{t!}\binom{r}{t}^{-1}$ for all $2 \leq t \leq r-1$. We remark that the smallest such r is 2 for $k=2$ and 4 for $k=4$, as proved in [6] and [4], respectively.

For general odd k we were not able to prove that the limit $\pi(r,t,k)$ exists, even when r is large. Nevertheless, arguments reminiscent of Theorem 1.2 yield that if k is odd, $t \geq 2$ and r is sufficiently large with respect to k and t, then $\limsup_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \leq \frac{1}{t!} \cdot \frac{2}{2\binom{r}{t}-1}$. We briefly sketch the proof idea.

A *t-tight component* is a collection of edges that can be ordered as $\{e_1, \ldots, e_m\}$ so that e_{i+1} shares at least t vertices with one of $e_1, \ldots, e_i$, for each $i \in [m-1]$. Given any k-free n-vertex r-graph $\mathcal{F}$, we let, for $i \in [2]$, $\mathcal{F}_i$ be the set of the edges of $\mathcal{F}$ which belong to components of size i, and $\mathcal{F}_3 := \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. For $i \in [2]$, define $\mathcal{G}_i$ to be the t-shadow of $\mathcal{F}_i$, and define $\mathcal{G}_3$ to be the collection of t-sets T such that $T \notin \mathcal{G}_1 \cup \mathcal{G}_2$ and there is a unique component in $\mathcal{F}_3$ that contains a 2-configuration whose vertex set contains T. With $\alpha := \frac{1}{2} \cdot \left(2\binom{r}{t} - 1\right)$, it is not hard to see that $|\mathcal{G}_i| \geq \alpha |\mathcal{F}_i|$ for $i \in [3]$. Therefore, since the sets $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ partition $\mathcal{F}$, it follows that $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \leq \frac{1}{\alpha}(|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3|) \leq \frac{1}{\alpha}\binom{n}{t}$, which implies the desired result.

We suspect this upper bound might be optimal, as this is the case for $k=3$ (see [4]).

Remark 6.1. *The recent work [5] mentioned in Remark 1.5 shows that, for $k=6$, the smallest r such that $\pi(r,t,6) = \frac{1}{t!}\binom{r}{t}^{-1}$ for all $2 \leq t \leq r-1$ is $r=4$. Moreover, it shows that the upper bound discussed above is optimal for $k \in \{5,7\}$.*

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