1 Proof of Mader's splitting theorem

Given a digraph D and a set of vertices X, let $d^+(X;D)$ be the number of edges from X to $\overline{X}$ (here $\overline{X} := V(D) \setminus X$), and similarly let $d^-(X;D)$ be the number of edges from $\overline{X}$ to X. We may omit D when it is clear what the underlying digraph is. For vertices x and y we denote by $\lambda(x,y;D)$ (or $\lambda(x,y)$ in short) the maximum number of edge-disjoint directed paths from x to y in D. By Menger's theorem, this is the minimum of $d^+(X)$ over all subsets X with $x \in X$ and $y \notin X$. For a vertex z in D, let $\lambda_z(D)$ be the minimum of $d^+(X)$, over all sets of vertices X such that $X \setminus \{z\},\ \overline{X} \setminus \{z\} \neq \emptyset$. Equivalently, this is the minimum value of $\lambda(x,y)$, over all pairs of distinct vertices x and y with $x, y \neq z$.

Given a directed edge $e = xy$ in D, write $\alpha(e) = x$ and $\varepsilon(e) = y$.

Theorem 1 (Mader [1])**.** *Let D be a digraph, let z be a vertex with $d^+(z) = d^-(z) \geq 1$, such that $\lambda_z(D) \geq k$. Then there are edges uz and zv such that the multidigraph D' obtained by splitting of the path uzv satisfies $\lambda_z(D') \geq k$.*

Given a digraph D, a vertex z and a set A such that $A \setminus \{z\},\ \overline{A} \setminus \{z\} \neq \emptyset$, we denote the contraction of D at A by D_A, and we will denote the vertex replacing A by a. We allow multiedges, so $d^+(A;D) = d^+(a;D_A)$. (Mader's paper also allows for loops.)

Observation 2. *Let D be a digraph, let z be a vertex, and let A be a set of vertices such that $z \notin A$ and $\overline{A} \setminus \{z\} \neq \emptyset$. If $d^+(A;D) = \lambda_z(D)$ then $\lambda_z(D_A) = \lambda_z(D)$.*

Proof. The observation follows by noting that for every subset $X \subseteq V(D) \setminus A$ we have $d^+(X;D_A) = d^+(X;D)$ and $d^+(X \cup \{a\};D_A) = d^+(X \cup A;D)$. □

Given a digraph D and edges h and k with $\varepsilon(h) = \alpha(k)$, we denote the digraph obtained by splitting the walk hk by D^{hk}.

Lemma 3. *Let D be a digraph, let z be a vertex with $d^+(z) = d^-(z)$, and let A be a set of vertices such that $z \notin A$ and $\overline{A} \setminus \{z\} \neq \emptyset$ satisfying $\lambda_z(D) = d^+(A)$. Suppose that h' is an edge in D_A directed towards z and k' is an edge in D_A directed from z, such that $\lambda_z((D_A)^{h'k'}) = \lambda_z(D_A)$. Then $\lambda_z(D^{hk}) = \lambda_z(D)$, where h and k are the edges of D that correspond to h' and k' in D_A.*

Proof. Write $m = \lambda_z(D)$, and $D' = D^{hk}$. Note that $D'_A = (D_A)^{h'k'}$.

Because $\lambda_z(D'_A) = \lambda_z((D_A)^{h'k'}) = d^+(a;D_A)$, at least one of $\alpha(h')$ and $\varepsilon(k')$ is not a. Equivalently, at least one of $\alpha(h)$ and $\varepsilon(k)$ is not in A. It follows that $d^+(B;D) = d^+(B;D')$ and $d^-(B;D) = d^-(B;D')$ for every $B \subseteq A$.

We will show that for every subset $X \subseteq V(D)$, that satisfies $X \setminus \{z\},\ \overline{X} \setminus \{z\} \neq \emptyset$, we have $d^+(X) \geq m$. We do so in three steps.

First, let $x \in A$ and $y \in \overline{A} \setminus \{z\}$. We claim that $\lambda(x,y;D') \geq m$. To see this, because $\lambda_z(D'_A) = m$, there are m pairwise edge-disjoint directed paths $P_1, \ldots, P_m$ in D'_A from a to y. Let P'_i be the path in D' that corresponds to P_i (namely, replace each edge e in P_i by the edge corresponding to e in D'). Because $\lambda_z(D) = m$, there are m pairwise edge-disjoint directed paths $Q_1, \ldots, Q_m$ in D from x to y. Since $d^+(A; D) = m$, each path Q_i contains exactly one edge from A to $\overline{A}$; denote it by e_i. Let Q_i' be the subpath of Q_i from x to $\alpha(e_i)$. Let R_i be the path obtained by concatenating P_i' and Q_i'. Then $R_1, \ldots, R_m$ are m pairwise edge-disjoint paths from x to y, implying that $\lambda(x, y; D') \geq m$, as claimed. By Menger's theorem, it follows that $d^+(X; D') \geq m$ for every $X \subseteq V(D)$ with $X \cap A \neq \emptyset$, and $\overline{X} \cap (\overline{A} \setminus \{z\}) \neq \emptyset$.

Second, if $X \cap A = \emptyset$ and $X \setminus \{z\} \neq \emptyset$, we have $d^+(X; D') = d^+(X; D_A') \geq m$, using $\lambda_z(D_A') = m$ for the inequality.

Finally, let X be a set of vertices such that $\overline{X} \cap (\overline{A} \setminus \{z\}) = \emptyset$ and $\overline{X} \setminus \{z\} \neq \emptyset$. In particular, $\overline{X} \subseteq A \cup \{z\}$. If $z \in X$, then $\overline{X} \subseteq A$, so, by a remark above and since $\lambda_z(D_A') = m$,

$$d^+(X; D') = d^-(\overline{X}; D') = d^-(\overline{X}; D_A') = d^+(V(D_A') \setminus \overline{X}; D_A') \geq m.$$

We now assume that $z \in \overline{X}$. Recall that there are m pairwise edge-disjoint paths $P_1, \ldots, P_m$ from A to $\overline{A} \setminus \{z\}$ in D'. Since $d^+(A; D') = m$, it follows that each path P_i contains exactly one edge from A to $\overline{A}$. Thus, if P_i contains an edge from A to z, this edge is followed by an edge from z to $\overline{A} \setminus \{z\}$. It follows that $e(A, z) \leq e(z, \overline{A} \setminus \{z\})$ (here $e(S, T)$ is the number of edges from S to T in D'). Hence $e(\overline{X} \setminus \{z\}, z) \leq e(A, z) \leq e(z, \overline{A} \setminus \{z\}) \leq e(z, X)$. Since $d^+(z) = d^-(z)$, we also have $e(X, z) \geq e(z, \overline{X} \setminus \{z\})$. It follows that $d^+(X \cup \{z\}; D') = d^+(X; D') - e(X, z) + e(z, \overline{X} \setminus \{z\}) \leq d^+(X; D')$. But $d^+(X \cup \{z\}; D') \geq m$, by the beginning of this paragraph, and so $d^+(X; D') \geq m$.

This completes the proof that $d^+(X; D') \geq m$ for every set of vertices X with $X \setminus \{z\},\ \overline{X} \setminus \{z\}) \neq \emptyset$. Thus $\lambda_z(D') = m$, as required. □

Proof of Theorem 1. Write $m := \lambda_z(D)$. The proof proceeds by induction on $|D|$. If $|D| \leq 3$, it is easy to check that the statement holds. Now suppose that $|D| \geq 4$. We may assume that there are edges e and f with $\varepsilon(e) = \alpha(f) = z$ and $\alpha(e) \neq \varepsilon(f)$. Indeed, otherwise, there is a vertex $y \neq z$ such that all edges incident with z are also incident with y (or are loops), and it follows that $\lambda_z(D \setminus \{z\}) = m$. Let e and f be such edges. If $\lambda_z(D^{ef}) \geq m$, we are done, so suppose otherwise. Then there exists a set of vertices A such that $A \setminus \{z\},\ \overline{A} \setminus \{z\} \neq \emptyset$ and $d^+(A; D^{ef}) < m$. Without loss of generality, $z \notin A$. Since $\lambda_z(D) = m$, we have $d^+(A; D) \geq m$. It follows that $d^+(A; D) = m$ and $\alpha(e), \varepsilon(f) \in A$, which implies that $|A| \geq 2$. Consider the graph D_A. It has fewer vertices than D, so by induction, there are edges h' and k' in this graph with $\varepsilon(h') = \alpha(k') = z$ and $\lambda_z((D_A)^{h'k'}) = m$. Let h and k be the edges in D corresponding to h' and k'. By Lemma 3, $\lambda_z(D^{hk}) = m$, as required. □

References

[1] W. Mader, *Konstruktion aller n-fach kantenzusammenhängenden Digraphen*, Europ. J. Combin. **3** (1982), 63–67.