

Monochromatic cycle partitions of 2-coloured graphs with minimum degree $3n/4$

Shoham Letzter

Department of Pure Mathematics and Mathematical Statistics

University of Cambridge

Wilberforce Road, Cambridge CB30WB, UK.

`shoham.letzter@math.ethz.ch`.

Abstract

Balogh, Barát, Gerbner, Gyárfás, and Sárközy made the following conjecture. Let G be a graph on n vertices with minimum degree at least $3n/4$. Then for every 2-edge-colouring of G , the vertex set $V(G)$ may be partitioned into two vertex-disjoint cycles, one of each colour.

We prove this conjecture for large n , improving approximate results by the aforementioned authors and by DeBiasio and Nelsen.

1 Introduction

1.1 History

While undergraduates in Budapest, Gerencsér and Gyárfás [11] proved the following simple result: for any 2-edge-colouring of the complete graph K_n , there exists a monochromatic path of length at least $\lceil 2n/3 \rceil$. It is easy to see that this statement is sharp. In their paper, Gerencsér and Gyárfás observe that a weaker result, asserting the existence of a monochromatic path of length at least $n/2$, can be deduced from the following simple observation: for any red and blue colouring of K_n , there is a Hamilton path which is the union of a red path and a blue path. The latter observation, simple as it is, inspired intensive research.

In a later paper, Gyárfás [12] proved that, in fact, more is true. He showed that for any red and blue colouring of K_n the vertices may be covered by a red cycle and a blue one sharing at most one vertex. Lehel went even further: he conjectured that for every 2-colouring of K_n the vertex set may be partitioned into two monochromatic cycles of distinct colours. We remark that in our context, the empty set, a single vertex and an edge are considered to be cycles. This conjecture first appeared in [2], where it was proved for some special colourings of K_n .

In 1998, almost twenty years after this conjecture was made, Łuczak, Rödl and Szemerédi [22] proved it for large n , using the regularity lemma. Ten years later, Allen [1] proved it for large n avoiding the use of the regularity lemma. Finally, Lehel’s conjecture was fully resolved in 2010 by Bessy and Thomassé [5] with an elegant and short proof.

1.2 Conjectures and progress

In the hope of generalising the above result of Gerencsér and Gyárfás, Schelp [26] considered 2-colourings of graphs which are not necessarily complete. In particular, he conjectured that given a graph G on n vertices with $\delta(G) > 3n/4$, there is a monochromatic path of length at least $2n/3$. Benevides, Łuczak, Skokan, Scott and White [4] and Gyárfás and Sárközy [16] proved approximate versions of this conjecture.

Inspired by Schelp’s conjecture, Balogh, Barát, Gerbner, Gyárfás, and Sárközy [3] proposed the following conjecture: given a graph G on n vertices with minimum degree $\delta(G) > 3n/4$, for every 2-colouring of G the vertex set can be partitioned into two monochromatic cycles of distinct colours. We note that there are examples of 2-coloured graphs G on n vertices with $\delta(G) = \lceil 3n/4 \rceil - 1$ which do not admit such a partition (see Section 2).

In their paper [3], the authors prove the following approximate version of their conjecture. For every $\varepsilon > 0$ there exists n_0 such that for every 2-coloured graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (3/4 + \varepsilon)n$, there exist vertex-disjoint monochromatic cycles of distinct colours covering all but at most εn of the vertices.

Recently, DeBiasio and Nelsen [8] proved the following stronger approximate version of the latter conjecture: for every $\varepsilon > 0$ there exists n_0 such that, for every 2-coloured graph G on $n \geq n_0$ vertices and $\delta(G) \geq (3/4 + \varepsilon)n$, the vertex set may be partitioned into two monochromatic cycles of distinct colours.

1.3 The main result

Our main result proves that the conjecture of Balogh et al. [3] holds if n is large enough.

Theorem 1. *There exists n_0 such that if a graph on $n \geq n_0$ vertices and minimum degree at least $3n/4$ is 2-coloured then its vertex set may be partitioned into two monochromatic cycles of different colours.*

In 1999, Łuczak [21] introduced a technique that uses the regularity lemma to reduce problems about paths and cycles into problems about connected matchings, which are matchings that are contained in a connected component. This technique, which we shall describe in more detail in Section 5, has become fairly standard and can be used to prove the approximate result of Balogh et al. [3]. The second result by DeBiasio and Nelsen [8], requires further ideas, most notably the

‘absorbing technique’ of Rödl, Ruciński and Szemerédi (see [25] and [20]). Nevertheless, the stronger conditions on the minimum degree make their proof a great deal easier than ours. In order to prove Theorem 1, we use a variety of additional ideas and techniques.

We remark that Theorem 1 is sharp. Indeed, for every $n \geq 4$, there exists a 2-coloured graph on n vertices with minimum degree $\lceil \frac{3n}{4} \rceil - 1$ admitting no partition into two monochromatic cycles of distinct colours. We give such extremal examples in the following section. These examples disprove the conjecture from [8], that a slightly stronger version of Theorem 1 may hold, namely that the conclusion holds for graphs with minimum degree at least $\frac{3n-3}{4}$.

The following section consists of some extremal examples for Theorem 1. In Section 3 we give an overview of the proof as well as the structure of the rest of this paper.

2 Sharpness

Before we turn to the proof of Theorem 1, we give some extremal examples showing that the theorem is sharp. More precisely, we give examples of 2-coloured graphs on n vertices with minimum degree $\lceil \frac{3n}{4} \rceil - 1$ admitting no partition into two monochromatic cycles of distinct colours. Figures 1, 2, 3a and 3b depict several families of such examples differing in the values of $n \pmod{4}$ for which they work.

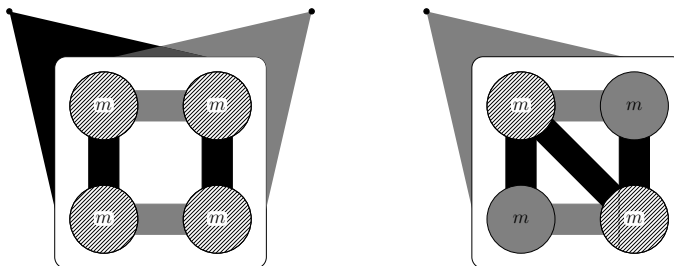


Figure 1: Examples of black and gray coloured graph on $n \in [4m - 1, 4m + 2]$ vertices, with minimum degree at least $\lceil \frac{3n}{4} \rceil - 1$, whose vertices cannot be partitioned into two disjoint monochromatic cycles of distinct colours.

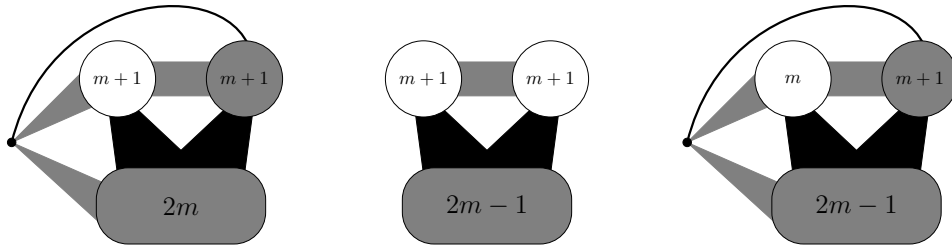


Figure 2: More examples

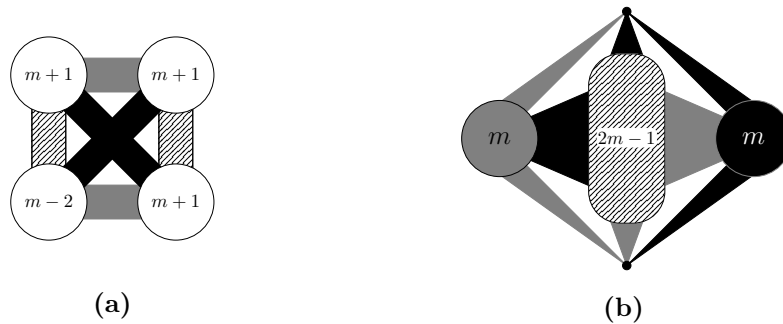


Figure 3: Even more examples

3 Overview

Suppose that G is a graph on n vertices and minimum degree $3n/4$ with a red and blue colouring of the edges. In what follows, we outline our proof that $V(G)$ may be partitioned into a red cycle and a blue one.

Given an edge coloured graph, a *monochromatic connected matching* is a monochromatic matching which is contained in a connected component of the same colour. Similarly to the earlier results on our problem (by Balogh et al. [3] and DeBiasio and Nelsen [8]), as well as many other results in the area, one of the key tools is the technique of reducing problems about cycles to problems about connected matchings using the regularity lemma. This technique was introduced by Łuczak [21] and since then has become fairly standard. In our setting, the basic idea, which is described in more detail in Section 5, is as follows. We are given a 2-coloured graph G and consider the reduced graph obtained by applying the regularity lemma. If the reduced graph has a perfect matching consisting of a red connected matching and a blue connected matching, we may use the blow-up lemma [17] (or in fact, a much simpler special case), to find two vertex-disjoint monochromatic cycles, a red one and a blue one, which cover almost all of the vertices.

The next ingredient is the ‘absorbing method’ of Rödl, Ruciński and Szemerédi (see [25] and [20]).

As in [8], in order to apply this method, we use a notion of ‘robust subgraphs’, which are defined to be graphs with certain expansion properties (see Section 4 for the exact definition). Such graphs can be shown to contain short ‘absorbing paths’, which are paths that can absorb small sets of vertices. We observe (see Section 5) that monochromatic connected components in the reduced graph (obtained from the regularity lemma) correspond to monochromatic robust subgraphs in the original graph. This observation allows us to obtain information about the rough structure, by using the regularity lemma and finding the corresponding robust subgraphs (see Section 7).

After a study of the some properties of the rough ‘robust structure’ of the graph, we aim to apply the regularity lemma again, in order to find a suitable perfect matching, namely a perfect matching consisting of a connected red matching and a connected blue matching. This matching is used to find two vertex-disjoint cycles, a red one and a blue one, which cover most of the vertices and have the additional absorbing property implying that the leftover vertices can be inserted into one of these cycles (see Sections 8 - 13).

We should like to emphasize that as we prove the sharp result, namely the minimum degree is at least $3n/4$, rather than $(3/4 + \varepsilon)n$ as in the previous results, new difficulties arise. Firstly, we have to deal with several different cases for the rough robust structure, some of which do not arise when $\delta(G) \geq (3/4 + \varepsilon)n$. Interestingly, these cases require a variety of ideas and techniques, making the proof of the general theorem rather intricate. Secondly, when applying the regularity lemma we cannot guarantee that the minimum degree would be at least $3n/4$, thus it may not be possible to find a suitable matching in the reduced graph directly.

The combination of the following two ideas helps us with these challenges. The first idea is to use stability versions of results promising a perfect matching. These enable us to conclude that if the reduced graph does not have the required perfect matching, it has some specific structure which we can further analyse to find the required monochromatic cycle partition.

The second idea is the following simple yet important observation: given two robust components of the same colour, if they can be connected with two vertex-disjoint paths, they may essentially be treated as one larger component. In several parts of the proof (see Sections 9, 11 and 12), we use this observation to conclude that either we may join two robust components to obtain a larger one, or the graph admits some restrictive structure, for which the desired partition may be found ‘by hand’ (see Sections 14 and 15). We remark that even at this stage, the proof is rather hard due to the fact that our result is sharp.

3.1 Notation

We use mostly standard notation. Write $|G|$ for the order of a graph G and $\delta(G)$ and $\Delta(G)$ for its minimum and maximum degrees respectively. The neighbourhood of a vertex $x \in V(G)$ is denoted by $N_G(x)$ and its degree by $d_G(x) = |N_G(x)|$. Given $A \subseteq V(G)$, we write $N_G(x, A) = N_G(x) \cap A$ and $d_G(x, A) = |N_G(x, A)|$. We will write, for example, $d(x, A)$ for $d_G(x, A)$ if this is unambiguous. Given a set of vertices $X \subseteq V(G)$, we write $G[X]$ for the graph induced by G on X . Similarly,

for disjoint subsets $X, Y \subseteq V(G)$, we write $G[X, Y]$ for the bipartite graph with bipartition $\{X, Y\}$ induced by G . We denote $e_G(X, Y) = |E(G[X, Y])|$.

Given a graph G , we denote a 2-colouring of G by $E(G) = E(G_B) \cup E(G_R)$, where G_B, G_R are graphs on vertex set $V(G)$ (in proper colouring, the graphs G_B, G_R are edge-disjoint). The edges of G_B are called blue edges and the edges of G_R are red edges. We sometimes use B or R for a subscript instead of G_B or G_R . For instance, $N_B(x)$ is a shorthand for $N_{G_B}(x)$.

We denote by $(u_1 u_2 \dots u_k)$ the path on vertices u_1, \dots, u_k taken in this order. We use the same notation to denote the cycle obtained by adding the edge (u_k, u_1) to the given path. It should be clear from the context if we are dealing with a path or a cycle. Given paths P_1, P_2 which share an end and are otherwise disjoint, we denote by $(P_1 P_2)$ the concatenation of the two paths. Similarly, if the paths share both ends but are otherwise disjoint, the same notation denotes the cycle obtained by joining the two paths.

Throughout this paper we omit floors and ceilings whenever the argument is unaffected. The constants in the hierarchies used to state our results are chosen from right to left. For example, the claim that a statement holds for $0 < a, \frac{1}{n} \ll b \ll c \ll 1$ means that there exist non-decreasing functions $f, g : (0, 1] \rightarrow (0, 1]$ and a constant c_0 such that the statement holds for all $0 < a, b, c \leq c_0$ and integers n with $b \leq f(c), a \leq g(b)$ and $n \geq \frac{1}{g(b)}$. We normally do not specify the functions in question.

3.2 Structure of the paper

In the next section, Section 4, we define our notion of robustness and prove some properties of robust components, most notably the existence of absorbing paths. In Section 5, we state the version of the regularity lemma that we use here. We also prove some results about the correspondence between connected components of the reduced graph and robust subgraphs of the original graph and describe the method of converting connect matchings in the reduced graph into cycles in the original graph. In Section 6, we list some results which will be used throughout the proof.

Sections 7 - 15 are devoted to the proof of Theorem 1. In Section 7, we obtain some information about the rough structure and point out how to prove Theorem 1 using the results in subsequent lemmas. In each of Sections 8 - 13, we consider one of the cases arising from the structural result in Section 7. These cases vary in difficulty and we have to use various techniques used to deal with them. In Sections 14 and 15, we prove Lemmas 29 and 30, which are used in earlier sections and prove the main theorem under certain restrictive conditions on the colouring and the structure of the graph. We conclude the paper in Section 16 with our final remarks.

4 Robust subgraphs

Similarly to the proof of DeBiasio and Nelsen [8], one of the main tools in our proof is the notion of robust subgraphs. As we shall see, these are graphs with certain expansion properties. The role

of robust subgraphs in our proof is similar to their role in [8], but our definition is different and is often easier to apply. Nevertheless, the two definitions are, in some sense, equivalent, as can be seen in Lemma 5.4 in [8]. After defining robust subgraphs, we state and prove some simple properties they possess. Finally, we prove that robust subgraphs contain ‘absorbing paths’, which may absorb small sets of vertices.

4.1 Definitions

We define two notions of robustness: strong and weak. The difference between the two is that strong robust subgraphs are far from being bipartite. It will be easier for our application, though not essential, to define a robust subgraph relative to a fixed ground graph. The precise definitions are as follows. Given a graph G , vertices $x, y \in V(G)$ and an integer l , denote by $\text{con}_{G,l}(x, y)$ the number of paths of length $l + 1$ in G between x and y .

Definition 2. Let G be a graph on n vertices.

A subgraph F of G is called (α, k) -strongly-robust if there exists $l \leq k$ such that for every pair of vertices x, y in F we have $|\text{con}_{F,l}(x, y)| \geq \alpha n^l$.

A subgraph F of G is called (α, k) -weakly-robust if there exists a partition $\{X, Y\}$ of $V(F)$ such that for some $l \leq k$ and every $x \in X, y \in Y$ we have $|\text{con}_{F',l}(x, y)| \geq \alpha n^l$, where $F' = F[X, Y]$.

As a shorthand, we often omit the parameters α and k when they are clear from the context. From now on, we use the term ‘robust’ to signify either strongly robust or weakly robust with suitable parameters. We point out that in our context α and k are fixed and n tends to infinity. We remark that an (α, k) -robust subgraph F of a graph G on n vertices has $\delta(F) \geq \alpha n$. In particular, robust subgraphs are always dense.

Before discussing some properties of robust subgraphs, let us give a few examples. Any graph of minimum degree at least $(1/2 + \alpha/2)n$ is $(\alpha, 1)$ -strongly-robust, because any two vertices have at least αn common neighbours. Similarly, the random graph $G(n, \alpha)$, is with high probability $(\alpha^2/2, 1)$ strongly robust. Furthermore, the blow-up of a path of length $k \geq 2$, where every vertex is replaced by a complete graph on n/k vertices, is $(\alpha, k - 1)$ -strongly-robust for a suitable α . Similarly, the blow-up of a path of length k , where the vertices are replaced by independent sets of size n/k , is an $(\alpha, k - 1)$ -weakly-robust graph.

4.2 Simple properties

We shall make use of some simple properties of robust graphs. The following lemma states that a robust subgraph remains robust after removing a small number of vertices.

Lemma 3. *Given $\alpha > 0$ and k an integer, the following holds for small enough β . Let G be a graph on n vertices and let F be an (α, k) -robust subgraph. Suppose that F' is obtained from F by removing at most βn vertices. Then F' is $(\alpha/2, k)$ -robust.*

Proof. We prove the lemma under the assumption that F is strongly robust; the proof in case F is weakly robust is analogous. Let $l \leq k$ satisfy $|con_{F,l}(x, y)| \geq \alpha n^l$ for every $x, y \in V(F)$. For every $x, y \in V(F)$, the number of paths of length $l + 1$ between x and y containing at least one vertex from $V(F) \setminus V(F')$ is at most $l\beta n^l \leq \frac{\alpha}{2}n^l$. It follows that $|con_{F',l}(x, y)| \geq \frac{\alpha}{2}n^l$ for every $x, y \in V(F)$, i.e. F' is $(\alpha/2, l)$ -strongly-robust. \square

The next lemma shows that a robust subgraph remains robust after removing a graph of small maximum degree.

Lemma 4. *Given $\alpha > 0$ and k , the following holds for suitably small β and large n . Let G be a graph on n vertices and let F be an (α, k) -robust subgraph. Suppose that F' is a subgraph of F such that for every vertex $v \in V(F)$ we have $\deg_{F'}(v) \geq \deg_F(v) - \beta n$. Then F' is $(\alpha/2, k)$ -robust.*

Proof. We prove this lemma for F strongly robust; the proof for F weakly robust is similar. Let $l \leq k$ be such that $|con_{F,l}(x, y)| \geq \alpha n^l$ for every $x, y \in V(F)$. Fix some $x, y \in V(F)$. We consider the family of paths in F of length $l + 1$ between x and y which contain at least one edge outside of F' . There are at most $l\beta n^l$ such paths, i.e. $|con_{F',l}(x, y)| \geq |con_{F,l}(x, y)| - l\beta n^l \geq \frac{\alpha}{2}n^l$ (for small enough β). It follows that F' is $(\alpha/2, l)$ -strongly-robust. \square

The following lemma states that a robust subgraph F remains robust after the addition of vertices which have a large neighbourhood in F .

Lemma 5. *Given $\alpha > 0$ and k an integer, the following holds for large enough n . Let G be a graph on n vertices and let F be an (α, k) -robust component. Let F' be a subgraph of G containing F , such that every vertex in $V(F') \setminus V(F)$ has at least αn neighbours in F . Then F' is $(\alpha^3/2, k + 2)$ -robust.*

Proof. We prove the statement assuming that F is strongly robust; the proof in case F' is weakly robust is very similar and we omit the details. Let $l \leq k$ satisfy $|con_{F,l}(x, y)| \geq \alpha n^l$ for every $x, y \in V(F)$. Fix some $x, y \in V(F')$. For every $z, w \in V(F)$ such that $z \in N(x)$ and $w \in N(y)$, we have $|con_{F,l}(w, z)| \geq \alpha n^l$. Thus, since every vertex in $V(F')$ has at least αn neighbours in F , we have that the number of walks between x and y in F' with $l + 2$ interior vertices is at least $\alpha^3 n^{l+2}$ for large n . Since there are at most $O(n^{l+1})$ such walks which are not paths, we have that $|con_{F',l}(x, y)| \geq \frac{\alpha^3}{2}n^{l+1}$. It follows that F' is $(\alpha^3/2, k + 2)$ -strongly-robust. \square

So far we listed and proved several simple properties of robust subgraphs. In the following subsection we state and prove a more interesting property.

4.3 Absorbing paths

The main reason robust subgraphs are so useful in our context, is the fact, which was proved by DeBiasio and Nelsen [8] that they contain short ‘absorbing paths’, which we define below. We conclude this section with a proof of this fact.

Definition 6. Let P be a path in a graph G . We say that P is a k -*absorbing path*, for every set W of at most k vertices, there is a path in G whose vertex set is $V(P) \cup W$ and whose ends are the ends of P .

Given a partition $\{X, Y\}$ of $V(G)$, we say that P is k -*weakly-absorbing* (with respect to the partition) if for every set W , which is disjoint of $V(P)$ and satisfies $|W \cap X| = |W \cap Y| \leq k/2$, there is a path whose vertex set is $V(P) \cup W$ and whose ends are the ends of P .

Lemma 7. Let $\frac{1}{n} \ll \rho \ll \alpha, \frac{1}{k} \ll 1$, let G be a graph on n vertices and let F be an (α, k) -robust subgraph of G . Then there exists a path Q in F , of length at most ρn , such that the following holds.

1. If F is strongly robust, then Q is $\rho^2 n$ -absorbing (in F);
2. If F is weakly robust, then Q is $\rho^2 n$ -weakly-absorbing (in F , with the bipartition of F).

The proof is an adaptation of DeBiasio and Nelsen's proof of Lemma 5.6 from [8]; we present the proof here as the terminology in [8] is quite different from ours. The main tool is absorbing method of Rödl, Ruciński and Szemerédi [25]. We shall use 'gadgets', which we will define to be Hamiltonian graphs that are can absorb a single vertex under the condition that it is adjacent to some of the vertices in the gadget. By a simple application of the probabilistic method and the robustness of the given graph, we show that there exists a not too large collection of vertex-disjoint gadgets, such that every vertex may be absorbed by a rather large number of them. From there it will be easy to construct the required path Q .

Proof of Lemma 7. We start by proving the first part of Lemma 7. Suppose that F is (α, k) -strongly-robust. In particular, $\delta(F) \geq \alpha n$ so we may apply the following claim.

Claim 8. Let $p, \frac{1}{n} \ll \alpha \ll 1$ and let F be a graph on at most n vertices with $\delta(F) \geq \alpha n$. Then there is a family \mathcal{F} of disjoint pairs of vertices of $V(F)$ with the following conditions.

- $|\mathcal{F}| \leq pn$;
- For every $u \in V(F)$ there are at least $\frac{1}{16}p\alpha n^2$ pairs $(x, y) \in \mathcal{F}$ such that x, y are adjacent to u .

Proof. Let \mathcal{F} be the family of pairs obtained by choosing each pair of vertices in $V(G)$ independently with probability $\frac{p}{n}$. By Chernoff's bound, we have that with high probability, the following properties hold.

- $|\mathcal{F}| \leq 2\frac{p}{n}\binom{n}{2} \leq pn$;
- For every $u \in V(F)$, \mathcal{F} contains at least $\frac{1}{2}\frac{p}{n}\binom{\alpha n}{2} \geq \frac{1}{8}p\alpha^2 n$ pairs (x, y) such that $x, y \in N(u)$.

The expected number of pairs of intersecting pairs in \mathcal{F} is at most $(\frac{p}{n})^2 n^3 \leq p^2 n$. It follows by Markov's inequality that with probability at least $1/2$, the number of pairs of intersecting pairs in \mathcal{F} is at most $2p^2 n$. In particular, we may pick a family \mathcal{F} which satisfies the above conditions and which has at most $2p^2 n$ pairs of intersecting pairs. We obtain a subfamily \mathcal{F}' of \mathcal{F} containing no intersecting pairs by deleting at most $2p^2 n$ pairs from \mathcal{F} . It is easy to verify that if p is suitably small, \mathcal{F}' satisfies the requirements of the claim. \square

Let $\mathcal{F} = \{(x_j, y_j)\}_{j=1}^N$ be a family of pairs as in Claim 8 (so $N \leq pn$). We use the following simple technical claim, to avoid divisibility issues.

Claim 9. *Let β be suitably small and n suitably large. Then for some $1 \leq l \leq k$, there are at least βn^{4l-2} paths of length $4l - 1$ between each pair of vertices in F .*

Proof. Since F is (α, k) -strongly-robust, there exists $l \leq k$ such that between every $u, v \in V(F)$ there are at least αn^{l-2} paths in F of length $l - 1$. We conclude that for every $u, v \in V(F)$ there are at least $\alpha^{10} n^{4l-2}$ walks of length $4l - 1$ between u and v . Indeed, given $u, v \in V(F)$, there are at least $(\alpha n)^6$ ways to pick edges $e_1, e_2, e_3 \in E(F)$, since we may pick one end of each edge in at least $|F| \geq \alpha n$ ways, and then there are at least αn ways to pick a neighbour. Denote $e_i = (a_i, b_i)$. There are $(\alpha n^{l-2})^4$ ways to pick paths P_1, P_2, P_3, P_4 in F of length $l - 1$ with ends u and a_1, b_1 and a_2, b_2 and a_3, b_3 and v respectively. It follows that there are at least $\alpha^{10} n^{4l-2}$ walks in F between u and v . At most $O(n^{4l-3})$ of them are not paths, so for large enough n , there are at least $\frac{\alpha^{10}}{2} n^{4l-2}$ walks of length $4l - 2$ between u and v . \square

We shall build vertex-disjoint paths Q_j of length $8l^2 - 4l + 1$ one by one for $j = 1, \dots, N$ as follows. Suppose that Q_1, \dots, Q_{j-1} are already defined. We would like to pick paths P and P_i as follows.

$$\begin{aligned} P &= (u_1, \dots, u_{4l}) \text{ a path with ends } x_j, y_j, \\ P_i &\text{ a path of length } 4l - 1 \text{ with ends } u_i, u_{i+3}, \text{ for } i = 1, 3, \dots, 4l - 5, \\ P_{4l-3} &\text{ a path of length } 4l - 1 \text{ with ends } u_{4l-3}, u_{4l-1}. \end{aligned}$$

It is easy to see, by the choice of l according to Claim 9, that if p is small enough, we may pick such paths to be vertex-disjoint of all previously defined paths and to have pairwise disjoint interiors. Define Q_j as follows.

$$Q_j = (u_2 u_1 P_1 u_4 u_3 P_3 u_6 \dots u_{4l-2} u_{4l-3} P_{4l-3} u_{4l-1} u_{4l}).$$

Suppose that $w \in V(G)$ is a neighbour of $x_j = u_1$ and $y_j = u_{2l}$. The following path is a path in F with vertex set $V(Q_j) \cup \{w\}$ and with the same ends as Q_j (this path is illustrated in Figure 4 together with Q_j).

$$(u_2 u_3 P_3 u_6 u_7 P_7 \dots u_{4l-5} P_{4l-5} u_{4l-2} u_{4l-1} P_{4l-3} u_{4l-3} u_{4l-4} P_{4l-7} \dots u_5 u_4 P_1 u_1 w u_{4l}).$$

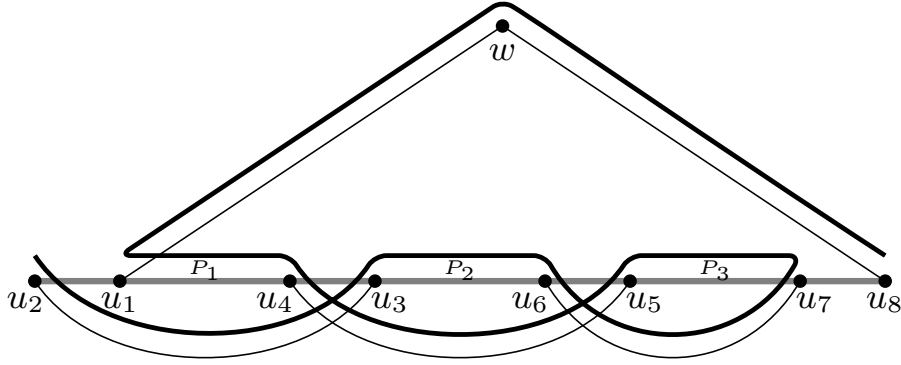


Figure 4: An illustration of the absorbing structure for $l = 2$. The path Q_j is represented by the straight line between u_2 and u_8 which is marked in grey, and the path absorbing w is represented by the bold black path.

Finally, we let Q be a path that contains Q_1, \dots, Q_N by connecting the ends of the Q_j 's with paths of length $4l - 1$. Denote $\rho = 8pk^2$, and note that we may pick p small enough such that $\rho^2 \leq \frac{1}{16}p\alpha^2$. The length of Q is at most ρn and for every vertex $z \in V(F) \setminus V(Q)$, there are at least $\rho^2 n$ values of $j \in [N]$ such that $x_j, y_j \in N(z)$. We show that Q has the desired absorbing property. Let W be a set of at most $\rho^2 n$ vertices in $V(F) \setminus V(Q)$ and denote $W = \{w_1, \dots, w_M\}$. We may pick distinct $j_1, \dots, j_M \in [N]$ such that $x_{j_i}, y_{j_i} \in N(w_i)$. Recall that for each $i \in [M]$, there is a path Q'_{j_i} in F on vertex set $V(Q_{j_i}) \cup \{w_i\}$ with the same ends as Q_{j_i} . By replacing the occurrence of Q_{j_i} by Q'_{j_i} in the path Q , we obtain a path on vertex set $V(Q) \cup W$ with the same ends as Q .

We now turn to the proof of the second part of Lemma 7. Let F be an (α, k) -weakly-robust subgraph with bipartition $\{X, Y\}$. The proof will use similar ideas, with some variations which take into account the bipartition of F . A similar argument as in Claim 8 implies that for small enough p we may find a family \mathcal{F} of disjoint quadruples of vertices of F with the following properties.

- $|\mathcal{F}| \leq pn$;
- For every $x \in X, y \in Y$ the number of quadruples (a, b, c, d) such that $a, c \in N(x)$ and $b, d \in N(y)$ is at least $\frac{1}{16}p\alpha^4 n$.

Denote $\mathcal{F} = \{(x_j, y_j, z_j, w_j)\}_{j=1}^{j=N}$. For $j = 1, \dots, N$ we pick a path Q_j as follows. As before, there exists $1 \leq l \leq k$ such that $|\text{con}_{F, 4l-2}(x, y)| \geq \beta n^{4l-2}$ for every $x \in X, y \in Y$. Assuming that Q_1, \dots, Q_{j-1} were already chosen to be paths of length at most $10l^2$, we may pick paths as follows, that are pairwise vertex-disjoint and disjoint of previously defined paths.

$$(a_1 b_1 \dots a_{2l} b_{2l}) \text{ a path with ends } a_1 = x_j, b_{2l} = y_j,$$

$$(c_1 d_1 \dots c_{2l} d_{2l}) \text{ a path with ends } c_1 = z_j, d_{2l} = w_j.$$

We may further pick paths P_i, R_i, S of length at most k as follows (as usual they may be chosen to be disjoint of previous paths).

$$\begin{aligned}
 P_i \text{ a path with ends } & \begin{cases} a_1, c_{2l} & \text{if } i = 1 \\ a_i, b_{i+1} & \text{if } 2 \leq i \leq 2l - 2 \\ a_{2l-1}, a_{2l} & \text{if } i = 2l - 1, \end{cases} \\
 R_i \text{ a path with ends } & \begin{cases} d_2, d_1 & \text{if } i = 2 \\ d_i, c_{i-1} & \text{if } 3 \leq i \leq 2l, \end{cases} \\
 S \text{ a path with ends } & c_1, b_2.
 \end{aligned}$$

Let

$$Q^j = (b_1 a_1 P_1 c_{2l} d_{2l} Q_{2l} c_{2l-1} \dots d_3 Q_3 c_2 d_2 Q_2 d_1 c_1 R b_2 a_2 P_2 b_3 \dots a_{2l-2} P_{2l-2} b_{2l-1} a_{2l-1} P_{2l-1} a_{2l} b_{2l}).$$

Suppose that u, v satisfy $x_j, z_j \in N(u)$ and $y_j, z_j \in N(v)$. Then the following path is a path in F with vertex set $V(Q_j) \cup \{u, v\}$ with the same ends as Q_j (See figure 5).

$$\begin{aligned}
 & (b_1 a_2 P_2 b_3 a_4 P_4 b_5 \dots a_{2l-2} P_{2l-2} b_{2l-1} a_{2l} P_{2l-1} a_{2l-1} b_{2l-2} P_{2l-3} a_{2l-3} \dots b_4 P_3 a_3 b_2 S c_1 x \\
 & a_1 P_1 c_{2l} d_{2l-1} R_{2l-1} c_{2l-2} \dots d_3 R_3 c_2 d_1 R_2 d_2 c_3 R_4 d_4 \dots c_{2l-1} R_{2l} d_{2l} y b_{2l}).
 \end{aligned}$$

We proceed as before to complete the proof of Lemma 7. □

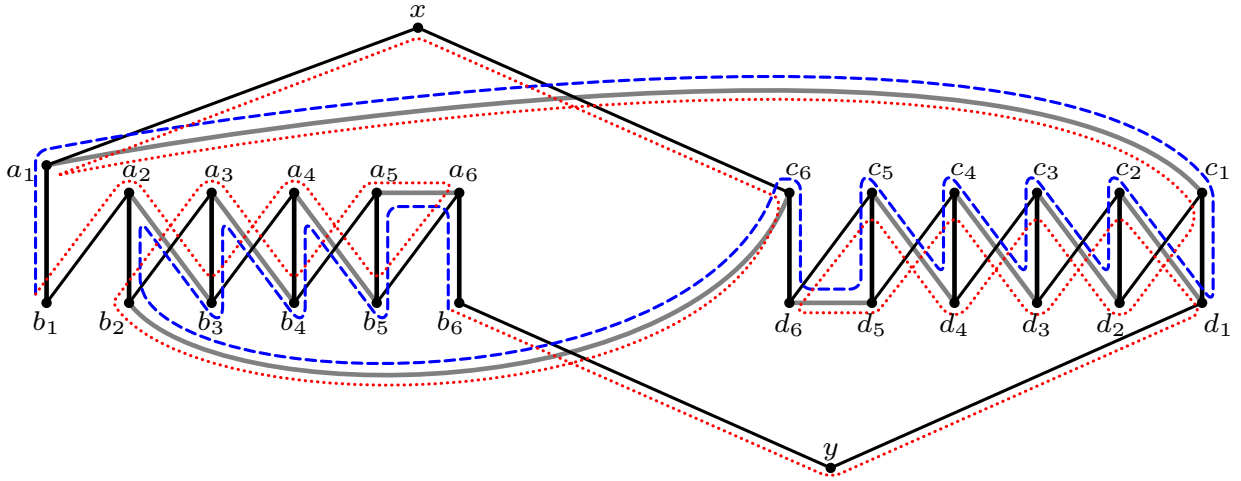


Figure 5: An illustration of the absorbing structure for $l = 3$. The black lines represent edges, whereas the grey ones represent paths. The dashed blue line represents the path Q^j and the dotted red one represents the path obtained from Q^j by absorbing w .

The proof of Lemma 7 concludes our introduction of the notion of robust subgraphs and their properties. In order to make use of the properties we established, we shall use Lemmas 12, 16 and 17 (stated below), which establish the connection between connected components of the reduced graph (given by the regularity lemma) and robust subgraphs.

5 The regularity lemma

In our proof, we shall use Szemerédi's regularity lemma extensively. We first introduce the relevant notation. Let U, W be disjoint subsets of vertices of a graph G . The density $d(U, W)$ of edges between U and W is defined to be $d(U, W) = \frac{e(U, W)}{|U||W|}$, where $e(U, W)$ is the number of edges between U and W . A bipartite graph with bipartition U, W is said to be ε -regular if for every $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$, the density $d(U', W')$ satisfies $|d(U', W') - d(U, W)| \leq \varepsilon$.

We use the so-called degree form of the regularity lemma (see [18]), adapted to a 2-coloured setting. Furthermore, it will be useful for our purpose to start with a cover of the vertices of a graph and require that the partition obtained by the lemma is a refinement of the initial cover.

Lemma 10. *For every $\varepsilon > 0$ and integer l there exists $M = M(l, \varepsilon)$ such that the following holds. Let G be a 2-coloured graph on n vertices, \mathcal{C} a cover of $V(G)$ with at most l parts and $d > 0$. Then there exists a partition $\{V_0, \dots, V_m\}$ of $V(G)$ and a subgraph G' of G with vertex set $V(G) \setminus V_0$, such that the following conditions hold.*

(R1) $m \leq M$;

(R2) Every V_i ($i \in [m]$) is contained in one of the parts of \mathcal{C} ;

(R3) $|V_0| \leq \varepsilon n$ and $|V_1| = \dots = |V_m| \leq \lceil \varepsilon n \rceil$;

(R4) $\deg_{G'}(v) \geq \deg_G(v) - (2d + \varepsilon)n$ for every $v \in V_1 \cup \dots \cup V_m$;

(R5) $e(G'[V_i]) = 0$ for $i \in [m]$;

(R6) All pairs (V_i, V_j) are ε -regular in both colours in G' , with density in each colour either 0 or at least d .

It is often useful to work with the reduced graph, obtained from the partition given by the regularity lemma as follows. Let $\{V_0, \dots, V_m\}$ be a partition of $V(G)$ and let G' be a subgraph of G , obtained by applying the above lemma to G . Define a graph Γ whose vertex set is $\{V_1, \dots, V_m\}$. A pair $V_i V_j$ is a t -coloured edge in Γ if it has density at least d in colour t in G' . Note that an edge of Γ can have more than one colour. We shall often use the following observation.

Observation 11. *Let G be a 2-coloured graph on n vertices with $\delta(G) \geq cn$ and let Γ be the (ε, d) -reduced graph obtained by applying Lemma 10. Then $\delta(\Gamma) \geq (c - 2d - \varepsilon)m$, where $m = |\Gamma|$.*

The rest of this section is divided into two parts. In Subsection 5.1, we establish the connection between robust subgraph and connected components of a reduced graph. In Subsection 5.2, we describe the interplay between connected matchings in a reduced graph and cycles in the original graph.

5.1 From connected components of the reduced graph to robust subgraphs

One of our main tools in the proof of Theorem 1 is the following lemma. It gives us the means to obtain a robust subgraph from a connected subgraph of the reduced graph.

Lemma 12. *Let $\alpha, \frac{1}{k}, \frac{1}{n} \ll \varepsilon, \frac{1}{l} \ll 1$ and $d \geq 4\varepsilon$. Let G be a graph on n vertices and let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 10. Suppose that Φ is a connected subgraph of Γ . Then there exists a subgraph F of G with the following properties.*

1. $V(F)$ contains at least $(1 - \varepsilon)$ of the vertices of each cluster in $V(\Phi)$;
2. F is (α, k) -robust. If Φ is bipartite, F is weakly robust, otherwise it is strongly robust.

In order to find the required robust component F , we consider the clusters represented by $V(\Phi)$, and for each of the clusters we remove vertices with low degree. The regularity of pairs of clusters which are connected by an edge, together with the choice of d , implies that the number of low degree vertices in each cluster is small. We show that the subgraph induced by the remaining vertices has the required expansion properties, using the regularity of the pairs of clusters.

Proof of Lemma 12. Denote $I = \{i \in [m] : V_i \in V(\Gamma)\}$, so $U = \bigcup_{i \in I} V_i$. For every $i \in I$, denote by I_i the set of indices $j \in I$ such that $V_i V_j \in E(\Phi)$ and let $N_i = \bigcup_{j \in I_i} V_j$. Let G' be the subgraph of G given by Lemma 10). Let

$$W_i = \{v \in V_i : \deg_{G'}(v, N_i) \leq 3\varepsilon|N_i|\}.$$

Claim 13. $|W_i| \leq \varepsilon|V_i|$ for every $i \in I$.

Proof. Suppose otherwise. Recall that for every $j \in I_i$, (V_i, V_j) is an ε -regular pair in G' with density at least 4ε . It follows that $d_{G'}(W_i, V_j) \geq d_{G'}(V_i, V_j) - \varepsilon \geq 3\varepsilon$. Hence, $e_{G'}(W_i, V_j) \geq 3\varepsilon|W_i||V_j|$ for $j \in I_i$ and $e_{G'}(W_i, N_i) \geq 3\varepsilon|W_i||N_i|$. It follows that there exists a vertex in W_i which is incident to at least $3\varepsilon|N_i|$ edges into N_i , a contradiction to the choice of W_i . \square

Define $W = \bigcup_{i \in I} W_i$ and $F = G_B[U \setminus W]$. Note that F satisfies Property 1 in Lemma 12. It remains to show that Property 2 holds. We suppose that Φ is non-bipartite, the proof for the bipartite case follows similarly. We use the following simple claim.

Claim 14. *Let G be a connected non-bipartite graph on n vertices. Then there exists $k \leq 3n$ such that between every two vertices of G there is a walk of length k .*

Proof. We first show that between every two vertices of G there is a walk of odd length not exceeding $3n$. Indeed, let $x, y \in V(G)$. Let C be an odd cycle, and pick some $z \in C$. Pick some path from x to z and a path from z to y . Combining the two paths, we obtain a walk from x to y of length at most $2n$. If this walk has even length, we add the cycle C to it. In any case we obtain an odd walk of length at most $3n$. Let k be the length of the longest of these walks. By possibly adding 2-cycles, we obtain walks of length k between each pair of vertices. \square

Fix some $k \leq 3n$ as in the previous claim. In the following claim, we show that the existence of a walk of length k in Φ implies the existence of many paths of the same length in F .

Claim 15. *Let $k \leq 3m$. There exists $\beta = \beta(l, \varepsilon)$ such that the following holds. Suppose that V_{i_1}, \dots, V_{i_k} is a walk in Φ . Let U_1 and U_k be subsets of $V_{i_1} \cap V(F)$ and $V_{i_k} \cap V(F)$, respectively, of size at least $2\varepsilon|V_1|$. Then there exists at least βn^{k+1} paths of length k in F between U_1 and U_k .*

Proof. Let $X_j = \{v \in V_{i_j} : \deg_H(v, V_{i_{j+1}}) \leq 3\varepsilon|V_1|\}$ for $j \in [k-1]$. As before it is easy to show that $|X_j| \leq \varepsilon|V_1|$. Let

$$Y_j = \begin{cases} U_1 \setminus X_1 & j = 1 \\ V_{i_j} \setminus (X_j \cup W_{i_j}) & j \in [2, k-1]. \end{cases}$$

Recall that $|W_j| \leq \varepsilon|V_1|$ for every j . It follows that for $j \in [k-1]$ and $v \in Y_j$, we have $\deg_H(v, Y_{j+1}) \geq \varepsilon|V_1|$. Thus there are at least $(\varepsilon|V_1|)^{k-2}$ walks of length $k-3$ between Y_1 and Y_{k-2} . Fix any such walk $(v = v_1, \dots, v_{k-2})$. Denote by A the neighbourhood of v_{k-2} in Y_{k-1} . Then $|A| \geq \varepsilon|V_1|$ and recall that $|U_k| \geq \varepsilon|V_1|$. Since $(V_{i_{k-1}}, V_{i_k})$ is an ε -regular pair in H with density at least 4ε , we have $e(A, U_k) \geq 3\varepsilon|A||U_k| \geq 3\varepsilon^3|V_1|^2$. Each such edge completes the above walk into a distinct walk in H between v_1 and U_k of length $k-1$.

We conclude that the total number of walks of length $k-1$ between U_1 and U_k is at least $\varepsilon^{k+1}|V_1|^k \geq \varepsilon^{k+1}(\frac{1-\varepsilon}{M})^k n^k$, where M is as in Lemma 10. Note that the number of such walks which are not paths (namely, a vertex appears more than once) is $O(n^{k-1})$. It follows that indeed, there are at least βn^k paths between U_1 and U_k of length $k-1$, where β is a suitable constant. \square

Let $x, y \in V(F)$ and suppose that $x \in V_i$ and $y \in V_j$. Note that the definition of F implies the existence of $t, s \in I$ such that $(V_i, V_t), (V_j, V_s) \in E(\Phi)$ and x and y have at least $2\varepsilon|V_1|$ neighbours in H in V_t and V_s respectively. It follows from Claim 15 that there are at least βn^k paths from the neighbourhood of x in V_t to the neighbourhood of y in V_s . This shows that $|\text{con}_{F,k}(x, y)| \geq \frac{\beta}{2} n^k$ for every $x, y \in V(F)$, implying that F is $(\beta/2, m)$ -strongly-robust, as required. \square

In fact, we need a stronger version of Lemma 12. In our application, since we deal with 2-coloured graph, we will typically have two collections of connected subgraphs of Γ , one for each colour, and it would be useful to obtain collections of robust subgraphs which preserve containment. For example, if in the reduced graph we have blue components Φ_1, Φ_2 and red components Φ_3, Φ_4 satisfying $V(\Phi_1) \cup V(\Phi_2) = V(\Phi_3) \cup V(\Phi_4)$, we would like the corresponding robust subgraphs

F_1, F_2, F_3, F_4 to satisfy the corresponding equality, namely $V(F_1) \cup V(F_2) = V(F_3) \cup V(F_4)$. This is achieved by the following lemma. It can be proved similarly to the previous lemma; we omit further details.

Lemma 16. *Let $\alpha, \frac{1}{k}, \frac{1}{n} \ll \varepsilon, \frac{1}{l} \ll 1$ and $d \geq 6\varepsilon$. Let G be a graph on n vertices with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 10 and let $\{V_0, \dots, V_m\}$ be the corresponding partition of $V(G)$. Let \mathcal{P}_B and \mathcal{P}_R be collections of disjoint connected subgraphs of Γ_B and Γ_R , respectively. Then there exist subsets $U_i \subseteq V_i$ satisfying the following properties.*

(i) $|U_i| \geq (1 - 2\varepsilon)|V_i|$ for $i \in [m]$;

(ii) Let $\Phi \in \mathcal{P}_t$, where $t \in \{B, R\}$ and denote $I = \{i \in [m] : V_i \in V(\Phi)\}$. Then the graph $F = G_t[\bigcup_{i \in I} U_i]$ is (α, k) -robust. If Φ is bipartite, F is weakly robust, otherwise it is strongly robust.

In our proof we shall find robust subgraphs and then apply the regularity lemma. Therefore we need the following result, stating that given a robust subgraph F in G , the corresponding subgraph of F in the reduced graph Γ is connected.

Lemma 17. *Let $\varepsilon, \frac{1}{n} \ll \alpha, \frac{1}{k}, \frac{1}{l} \ll 1$ and let G be a graph on n vertices with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F is an (α, k) -robust subgraph of G_t , where $t \in \{B, R\}$ and let \mathcal{C} be a cover of $V(G)$ with at most l parts refining $\{V(F), V(G) \setminus V(F)\}$. Let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 10. Then the t -coloured subgraph Φ of Γ spanned by the clusters contained in $V(F)$ is connected.*

Proof. Let G' be the corresponding subgraph of G obtained by applying Lemma 10. Let U be the set of vertices of F which belong to sets V_i contained in $V(F)$. Denote $F' = G'_t[U]$. Recall that by Lemma 10, we have that $|V(F) \setminus U| \leq \varepsilon n$ and $\deg_{G'}(v) \geq \deg_G(v) - 9\varepsilon n$. In particular $\deg_{F'}(v) \geq \deg_F(v) - 9\varepsilon n$ for every $v \in V(F')$. Hence F' is obtained by removing at most εn vertices of F and then removing a subgraph with maximum degree at most $9\varepsilon n$. By Lemmas 3 and 4, we have that F' is $(\alpha/4, k)$ -robust in G' . In particular, F' is connected and it follows that Φ is connected. \square

5.2 From connected matchings to long cycles

We shall use the technique of converting connected matchings in the reduced graph into cycles in the original graph. This was introduced by Łuczak [21], and since then has become fairly standard (see [3], [8] and [14], [15], [16] and [22]). We shall use the following lemma, which we state here without proof.

Lemma 18. *Let $\varepsilon > 0$ and $d \geq 3\varepsilon$ and let n be suitably large. Let G be a graph on n vertices and let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 10. Suppose that \mathcal{M} is a connected matching in Γ and denote by U the set of vertices spanned by the clusters of \mathcal{M} . Then G contains a cycle covering at least $(1 - 6\varepsilon)|U|$ of the vertices of U .*

In fact, we shall need a slight generalisation of Lemma 18. The usual setting in which we apply the described technique is as follows. We consider the reduced graph obtained by applying the regularity lemma. In the reduced graph, we find a perfect matching consisting of a connected blue matching and a connected red matching, and we use the described technique to find two disjoint cycles, one blue and one red, which together cover almost all of the vertices. This two coloured version of Lemma 18 can be proved similarly, but the constants change slightly: we need $d \geq 6\varepsilon$ and are able to cover $(1 - 9\varepsilon)n$ of the vertices of each cluster in the matchings.

Finally, we point out that we often first find ‘absorbing paths’ in the original graph and then apply the regularity lemma to the remaining graph. When building the cycles obtained by the connected matchings, we would like them to contain these predefined paths. This is obtained by the same method, enabling us to find a path (rather than a cycle) between the neighbourhoods of the two ends of the absorbing path.

This concludes the introduction of the tools we shall need for our proof of Theorem 1. To complete the preliminary material needed for our proof, we list several extremal results in the next section.

6 Extremal results

In this section we list extremal results that we shall use in our proofs. They concern mainly with the existence of matchings, paths and cycles in graphs with certain structural conditions.

The following is Chvátal’s theorem [7] giving sufficient conditions on the degree sequence of a graph for containing a Hamilton cycle.

Theorem 19 (Chvátal [7]). *Let G be a graph on $n \geq 3$ vertices and let $d_1 \leq \dots \leq d_n$ be the degree sequence of G . Suppose that $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for every $i \leq n/2$. Then G contains a Hamilton cycle.*

In some cases it is easier to use the following version of Chvátal’s result for bipartite graphs.

Corollary 20. *Let G be a balanced bipartite graph on $2n$ vertices with bipartition $\{X, Y\}$. Let $x_1 \leq \dots \leq x_n$ be the degree sequence of X and let $y_1 \leq \dots \leq y_n$ be the degree sequence of Y . Suppose that $x_i \geq i + 1$ or $y_{n-i} \geq n - i + 1$ for every $i \in [n]$. Then G contains a Hamilton cycle.*

Proof. Consider the graph G' obtained from G by adding all edges with both ends in X . By Theorem 19, G' contains a Hamilton cycle C . As $|X| = |Y|$, the cycle C contains no edges with both ends in X , i.e. C is a Hamilton cycle in G . \square

The following is a simple result by Erdős and Gallai [9], giving an upper bound on the number of edges in a graph with no path of a given length.

Theorem 21 (Erdős, Gallai [9]). *Let G be a graph on n vertices with no paths of length at least $l + 1$, then $e(G) \leq nl/2$.*

A graph on n vertices is called *pancyclic* if for every $l \leq n$, G contains a cycle on l vertices. The following result by Bondy [6] is a generalisation of Dirac's Theorem, asserting that graphs with large enough minimum degree are pancyclic.

Theorem 22 (Bondy [6]). *Let G be a graph on n vertices with $\delta(G) > n/2$. Then G is pancyclic.*

In the following two subsections, we use the well known theorem of Tutte, which gives a necessary and sufficient condition for having a perfect matching.

Theorem 23 (Tutte). *Let G be a graph on an even number of vertices. Then G has a perfect matching if and only if for every set of vertices U , the number of odd components of $G \setminus U$ is at most $|U|$.*

6.1 Matchings in tripartite graphs

In section 13 we shall analyse conditions for certain tripartite graph to have a perfect matching. Here we describe the extremal results we shall need for the analysis. We use a stability version of the following lemma of DeBiasio and Nelsen [8].

Lemma 24 (DeBiasio, Nelsen [8]). *Let n be even, and let G be a tripartite graph on n vertices with tripartition $\{X_1, X_2, X_3\}$. Suppose that $|X_i| \leq n/2$ and $\deg(x) > 3n/4 - |X_i|$ for every $x \in X_i$, $i \in [3]$. Then G has a perfect matching.*

The following lemma is a stability version of Lemma 24. We prove it by applying Tutte's theorem (Theorem 23).

Lemma 25. *Let $\frac{1}{n} \ll \varepsilon \ll 1$, where n is even. Let G be a graph on n vertices with tripartition $\{X_1, X_2, X_3\}$ such that $|X_i| \leq (1/2 - 4\varepsilon)n$ and $\deg(x, V(G) \setminus X_i) \geq (3/4 - \varepsilon)n - |X_i|$ for $i \in [3]$ and $x \in X_i$. Then one of the following holds.*

1. G has a perfect matching;
2. There is an independent set Y such that $Y \subseteq X_i \cup X_j$ and $|Y \cap X_i|, |Y \cap X_j| \geq (1/4 - 5\varepsilon)n$ for some $1 \leq i < j \leq 3$.

Proof. We assume that G has no perfect matching. By Tutte's theorem, Theorem 23, there exists a subset $S \subseteq V(G)$ such that the number of odd components of $G \setminus S$ is larger than $|S|$.

Denote $G' = G \setminus S$. We show first that $\delta(G') \leq \varepsilon n$. Suppose not. Then the number of components of G' is at most $1/\varepsilon$, thus $|S| \leq 1/\varepsilon$. We show that G' is connected, contradicting the choice of S . Given $u, v \in X_i$, they have at most $(1/4 + \varepsilon)n$ non-neighbours in $V(G) \setminus X_i$. But $|V(G) \setminus X_i| \geq (1/2 + 4\varepsilon)n$, hence u, v have at least $2\varepsilon n \geq 1/\varepsilon$ common neighbours. It follows that indeed, G' is connected, so $|S| \geq \varepsilon n$. By the assumptions on G , $\delta(G) \geq (3/4 - \varepsilon)n - \max\{|X_1|, |X_2|, |X_3|\} \geq (1/4 + 3\varepsilon)n$. Since $\delta(G') \leq \varepsilon n$, we have $|S| \geq (1/4 + 2\varepsilon)n$.

Note that $|S| \leq n/2$, because the number of odd component of G' is at most $n - |S|$. Denote $X'_i = X_i \setminus S$. We show that $|X'_i| \leq (1/4 + 2\varepsilon)n$ for $i \in [3]$. Indeed, suppose that $|X'_1| \geq (1/4 + 2\varepsilon)n$. Then for every vertex $u \in X'_2 \cup X'_3$ we have $\deg_{G'}(u) \geq \varepsilon n$. In particular, u is in a component of G' order at least εn . Furthermore, every non-isolated vertex of X'_1 is adjacent to some vertex in $X'_2 \cup X'_3$, and thus is in a component of size at least εn . Since $X'_2 \cup X'_3$ is non-empty (e.g. because $|S| \leq n/2$), it follows that G' has at most $(1/4 + \varepsilon)n$ isolated vertices, and the rest of the vertices are in components of order at least εn . Hence the number of odd components of G' is at most $(1/4 + \varepsilon)n + 1/\varepsilon \leq |S|$, a contradiction.

Denote $S_i = X_i \cap S$, $i \in [3]$. Consider the three quantities $|X_1| + |S_2| + |S_3|$, $|X_2| + |S_1| + |S_3|$, $|X_3| + |S_1| + |S_2|$. Their sum is $n + 2|S| \leq 2n$. Without loss of generality, it follows that $|X_1| + |S_2| + |S_3| \leq 2n/3$. Thus for every $u \in X'_1$,

$$\deg_{G'}(u) \geq (3/4 - \varepsilon)n - |X_1| - |S_2| - |S_3| \geq (1/12 - \varepsilon)n.$$

If, in addition, $|X'_1| + |X'_2| \geq (1/4 + 2\varepsilon)n$, then every vertex in X'_3 is in a component of order at least εn . But the number of isolated vertices in X'_2 is at most $(1/4 + \varepsilon)n$, so the number of odd components in G' is at most $(1/4 + \varepsilon)n + 1/\varepsilon \leq |S|$. We conclude that $|X'_1| + |X'_2| \leq (1/4 + 2\varepsilon)n$. Recall that $|X'_3| \leq (1/4 + 2\varepsilon)n$, implying that $|G'| \leq (1/2 + 4\varepsilon)n$, i.e. $|S| \geq (1/2 - 4\varepsilon)n$.

If $X'_1 \neq \emptyset$, G' contains a component of order at least $(1/12 - \varepsilon)n$ in G' , so there are at most $(5/12 + 5\varepsilon)n + 1 \leq |S|$ components in G' , a contradiction. The set Y , obtained by picking one vertex from each component of G' , is an independent set in $X'_2 \cup X'_3$ of size at least $|S|$. Let $Y_i = Y \cap X'_i$. Then there are no $Y_2 - Y_3$ edges, implying that $|Y_2|, |Y_3| \leq (1/4 + \varepsilon)n$, and so $|Y_2|, |Y_3| \geq (1/4 - 5\varepsilon)n$. \square

6.2 Matchings in bipartite graphs

In Section 13, we shall also need the following stability result for graphs with a bipartition satisfying certain conditions. The proof is again an application of Theorem 23.

Lemma 26. *Let $\frac{1}{n} \ll \varepsilon \ll 1$, where n is even. Let G be a graph on n vertices, and suppose that $\{X_1, X_2\}$ is a partition of $V(G)$ such that $|X_i| \geq (1/2 - \varepsilon)n$ and $\deg(u, X_{3-i}) \geq (3/4 - \varepsilon)n - |X_i|$ for $i \in [2]$, $u \in X_i$. Then one of the following conditions holds.*

1. G has a perfect matching;

2. G is not 2-connected;
3. There exists an independent set $A_i \subseteq X_i$ (for some $i \in [2]$) of order at least $(1/4 - 4\varepsilon)m$ such that $|N(A_i)| \leq (1/4 + 3\varepsilon)n$;
4. $|X_i| > |X_{3-i}|$ and X_i contains an independent set of size at least $(1/2 - \varepsilon)n$;
5. There is an independent set A satisfying $|A| \geq (1/2 - 6\varepsilon)n$ and $|A \cap X_i| \geq (1/4 - 9\varepsilon)n$ for $i \in [2]$.

Proof. Suppose that G has no perfect matching. It follows from Tutte's theorem that there is a set S such that the number of odd components in $G' = G \setminus S$ is larger than $|S|$. Denote $S_i = S \cap X_i$ and $X'_i = X_i \setminus S$.

If $\delta(G') \geq \varepsilon n$, the number of components of G' is at most $1/\varepsilon$, implying that $|S| \leq 1/\varepsilon$. It follows that G' consists of at most two connected components, implying that $|S| \leq 1$, so G is not 2-connected, as required. We now assume that $\delta(G') \leq \varepsilon n$, so $|S| \geq (1/4 - 2\varepsilon)n$.

Suppose that X'_1 and X'_2 are non-empty and $|X'_1|, |X'_2| \leq (1/4 + 3\varepsilon)n$, so $|S| \geq (1/2 - 6\varepsilon)n$. Then G' has at least $|S|$ connected components, implying that there is an independent set $A \subseteq X'_1 \cup X'_2$ of size at least $(1/2 - 6\varepsilon)n$. It follows that $|A \cap X'_i| \geq (1/4 - 9\varepsilon)n$, and thus the fifth condition holds.

Suppose now that X'_1 and X'_2 are both non-empty and $|X'_1| \geq (1/4 + 3\varepsilon)n$. Denote by A_1 the set of isolated vertices in X'_1 . Then every vertex in $X'_2 \cup (X'_1 \setminus A_1)$ belongs to a component of G' of size at least εn . Furthermore, $|A_1| \leq (1/4 + 2\varepsilon)n$. It follows that the number of components is at most $|A_1| + 1/\varepsilon \leq (1/4 + 3\varepsilon)m$, implying that $|S| \leq (1/4 + 3\varepsilon)m$. But $|S| \geq (1/4 - 2\varepsilon)n$, so $|A_1| \geq |S| - 1/\varepsilon \geq (1/4 - 3\varepsilon)n$. Since $N(A_1) \subseteq S$, the third condition holds.

It remains to consider the case where, say, $X'_2 = \emptyset$, so $X_2 \subseteq S$, implying that $|S| \geq (1/2 - \varepsilon)n$. Thus $G[X'_1]$ consists of at least $(1/2 - \varepsilon)n$ components, in particular it contains an independent set of size at least $(1/2 - \varepsilon)n$, and the fourth condition holds. \square

6.3 Hamilton cycles in bipartite graphs

The following result is a stability version of a special case of Corollary 20, which we shall use in Section 14, for the proof of Lemma 29 below.

Lemma 27. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a balanced bipartite graph on n vertices with bipartition $\{X_1, X_2\}$. Suppose that $\delta(G) \geq (1/4 - \varepsilon)n$ and between every two subsets of X_1 and X_2 of size at least $(1/4 - 3\varepsilon)n$ there are at least εn^2 edges. Then G is Hamiltonian. Furthermore, there is a Hamilton path between every pair of points x_1, x_2 where $x_1 \in X_1$ and $x_2 \in X_2$.*

We prove this result by a relatively simple application of the absorbing method of Rödl, Ruciński and Szemerédi [22]. In fact, all we need for this proof is Lemma 7, asserting the existence of short absorbing paths in robust subgraphs, and the regularity lemma. A graph G as in Lemma 27 is

$(\varepsilon, 2)$ -weakly-robust, thus by Lemma 7, it is possible to find an absorbing path P in G . We consider the reduced graph Γ , obtained from applying the regularity lemma on the graph $G \setminus V(P)$. We deduce from the conditions of the lemma that Γ has an almost perfect matching, implying that G contains a cycle extending P and spanning almost all vertices. The remaining vertices may be absorbed by P .

Proof of Lemma 27. It is easy to check from the conditions that G is $(\varepsilon, 2)$ -weakly-robust with bipartition $\{X_1, X_2\}$. It follows from Lemma 7, that for $\rho > 0$ is small enough, there exists a $\rho^2 n$ -absorbing path P in G of length at most ρn . Namely, if $W \subseteq V(G) \setminus V(P)$ is such that $|W \cap X_1| = |W \cap X_2| \leq \rho^2 n$, then $G[V(P) \cup W]$ contains a Hamilton path with the same ends as P . Note that we may assume for convenience that P has one end in X_1 and the other in X_2 .

Pick $\eta > 0$ suitably small. Apply the regularity lemma, Lemma 10, with the graph $G \setminus V(P)$, the bipartition $\{X'_1, X'_2\}$, where $X'_i = X \setminus V(P)$ and parameter η . Let G' be the subgraph of G promised by Lemma 10 and let $\{V_0, \dots, V_m\}$ be the given partition. We may assume that V_0 has the same number of vertices in X_1 and X_2 . Let Γ be the corresponding $(\eta, 4\eta)$ -reduced graph. Then Γ is a balanced bipartite graph. Let $m = |\Gamma|$.

Claim 28. Γ has a perfect matching.

Proof. Note that Γ has minimum degree at least $(1/4 - 2\varepsilon)m$ (using Observation 11 and assuming that η is sufficiently small). Furthermore, if $\{U_1, U_2\}$ is the bipartition of Γ , then there is an edge between W_1 and W_2 for every choice of sets $W_i \subseteq U_i$ of size at least $(1/4 - 2\varepsilon)m$ (this follows from the assumption on G). Suppose that Γ has no perfect matching. Then, by Hall's theorem, there is a set S in U_1 for which $|N(S)| < |S|$. But $|N(S)| \geq (1/4 - 2\varepsilon)m$ (by the minimum degree condition), so $|S| > (1/4 - 2\varepsilon)m$. Similarly, $|S| \leq (1/4 + 2\varepsilon)m$ (otherwise, every vertex in U_2 has a neighbour in S). It follows that the set T , defined as $T = U_2 \setminus N(S)$, has size at least $(1/4 - 2\varepsilon)m$. But then there are no $S - T$ edges in Γ , a contradiction. \square

Consider a perfect matching M in Γ . Then M is a connected matching as Γ is connected. By Lemma 18, we obtain a cycle C in G , containing the path P and spanning all but at most $6\eta n \leq \rho^2 n$ vertices. Denote $W = V(G) \setminus V(C)$. Since G is a balanced bipartite graph, we have $|W \cap X_1| = |W \cap X_2| \leq \rho^2 n$. It follows from the absorbing property of P that the vertices of W may be absorbed into P and thus into C to obtain a Hamilton cycle. It is easy to modify the proof to obtain a Hamilton path between any given vertices $x_i \in X_i$. \square

6.4 Monochromatic cycle partitions in 2-coloured graphs with the red graph almost disconnected

The following two lemmas, Lemmas 29 and 30, state that a 2-coloured graph G has the desired partition into a red cycle and a blue one if G admits some restrictive structural property. We shall

use these results several times in the following sections and delay their proofs to the ends of the paper. In Sections 14 and 15 we Lemmas 29 and 30 respectively.

Lemma 29. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that $S, T \subseteq V(G)$ satisfy the following conditions.*

- S, T are disjoint and $|S|, |T| \geq (1/2 - \varepsilon)n$;
- $\delta(G_B[S, T]) \geq (1/4 - \varepsilon)n$;
- For every $S' \subseteq S, T' \subseteq T$ with $|S'|, |T'| \geq (1/4 - 100\varepsilon)n$, we have $e(G_B[S', T']) \geq 25\varepsilon n^2$.

Then $V(G)$ may be partitioned into a red cycle and a blue one.

Lemma 30. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that there exists a partition $\{S, T, X\}$ of $V(G)$ with the following properties.*

- $|S|, |T| \geq (1/2 - \varepsilon)n$;
- $|X| \leq 2$ and if $|X| = 2$, there exists $u \in X$ such that $\deg_R(x, S) \leq \varepsilon n$ or $\deg_R(x, T) \leq \varepsilon n$;
- The sets S and T belong to different components of $G_R \setminus X$.

Then $V(G)$ may be partitioned into a red cycle and a blue one.

This concludes the preliminary material needed for the proof of Theorem 1. We are now finally ready to turn to the heart of the proof.

7 Rough structure

In this section we make the first step towards our proof of Theorem 1. We use the regularity lemma, Lemma 10, to obtain information about the rough structure.

Lemma 31. *Let $\frac{1}{n} \ll \alpha, \frac{1}{k} \ll \varepsilon \ll 1$ and let G be a graph with $\delta(G) \geq 3n/4$. Let $E(G) = E(G_B) \cup E(G_R)$ be a 2-colouring of G . Then one of the following assertions holds, where a robust component refers to an (α, k) -robust subgraph, possibly with the roles of red and blue reversed.*

1. *There exists a monochromatic strongly robust blue subgraph on at least $(1 - \varepsilon)n$ vertices.*
2. *There exists a weakly robust blue subgraph of order at least $(1 - \varepsilon/4)n$ and a red strongly robust subgraph of order at least $(1/2 + \varepsilon/2)n$.*

3. There exists a weakly robust blue component with bipartition $\{X_1, X_2\}$ where $|X_1|, |X_2| \geq (1/2 - \varepsilon)n$ and for each $i \in [2]$, $e(G_B[X_i]) \leq \varepsilon n^2$ and one of the following holds.

- (a) $G_R[X_i]$ is strongly robust;
- (b) $G_R[X_i]$ is weakly robust with bipartition $\{Y_{i,1}, Y_{i,2}\}$ satisfying $|Y_{i,j}| \geq (1/4 - \varepsilon)n$ and $e(G_R[Y_{i,j}]) \leq \varepsilon n^2$;
- (c) There exists a partition $\{Y_{i,1}, Y_{i,2}\}$ of X_i such that $|Y_{i,j}| \geq (1/4 - \varepsilon)n$, $G_R[Y_{i,j}]$ is strongly robust and $e(G_R[Y_{i,1}, Y_{i,2}]) \leq \varepsilon n^2$.

Furthermore, if Condition 3b holds for $i = 1, 2$, then, in addition, $e(G_R[Y_{1,j}, Y_{2,j}]) \leq \varepsilon n^2$ for $j \in [2]$.

4. There exist a blue strongly robust subgraph and a red robust subgraph, each has order at least $(3/4 - \varepsilon)n$ and together they span all but at most εn of the vertices.

5. There exist sets X_1, X_2, Y_1, Y_2 of order at least $(1/2 - \varepsilon)n$ such that

- (a) X_1, X_2 are disjoint, Y_1, Y_2 are disjoint and $X_1 \cup X_2 = Y_1 \cup Y_2$;
- (b) $|X_i \cap Y_j| \geq (1/4 - \varepsilon)n$ for $i, j \in [2]$;
- (c) $G_B[X_i]$ is robust and $G_R[Y_i]$ is strongly robust.

We remark that in light of the variety of extremal examples for Theorem 1 (see Section 2), it should not be surprising that there is a large number of cases to consider for the rough structure. Furthermore, it is perhaps useful to note that many of the above cases describe the structure of the extremal examples we gave in Section 2. For example, the left-hand graph in Figure 1 corresponds to Condition 5, Figures 2 and 3a correspond to Condition 3 and Figure 3b corresponds to Condition 4.

This lemma, technical as it seems, is a simple application of Lemmas 12 and 16, which imply that monochromatic components in the reduced graph correspond to robust subgraphs in the original graph. After applying the regularity lemma, we obtain a reduced graph Γ , which has minimum degree close to $3m/4$ where $m = |\Gamma|$. It is a routine check to verify that either there is a spanning monochromatic component, or there are two monochromatic components of size almost $3m/4$ spanning the whole vertex set, or for each colour there are two almost half-sized components spanning the whole vertex set. In the case where there is a bipartite spanning monochromatic component, further analysis is needed to show that one of Conditions 1, 2, 3 holds.

Proof of Lemma 31. Set $\eta = 48\varepsilon$. Let Γ be a $(\eta, 6\eta)$ -reduced graph obtained by applying Lemma 10 to the graph G . Note that $\delta(\Gamma) \geq (3/4 - 13\eta)m$, where $m = |\Gamma|$. Without loss of generality, we assume that the largest monochromatic component is blue and denote it by Φ_1 .

Suppose first that Φ_1 is a spanning subgraph of Γ . If it is non-bipartite, by Lemma 12, there is a strongly robust blue subgraph F_1 of order at least $(1 - 2\eta)n$, as in Condition 1.

Thus we assume that Φ_1 is bipartite, with bipartition $\{X_1, X_2\}$ where $|X_1| \geq |X_2|$. If $|X_1| > (1/2 + 26\eta)m$, then $\delta(\Gamma[X_1]) \geq (3/4 - 13\eta)m - |X_2| > |X_1|/2$. It follows that $\Gamma[X_1]$ is a red non-bipartite component. By Lemma 16, we obtain a weakly robust blue subgraph F_1 on at least $(1 - 3\eta)n$ vertices and a red strongly robust subgraph F_2 on at least $(1/2 + 23\eta)n$ vertices, as in Condition 2.

We assume now that $|X_1| \leq (1/2 + 26\eta)m$. Then $(1/2 - 26\eta)m \leq |X_1|, |X_2| \leq (1/2 + 26\eta)m$. Denote $\Gamma_i = \Gamma[X_i]$ for $i \in [2]$. Γ_i contains only red edges and $\delta(\Gamma_i) \geq (1/4 - 39\eta)m$. Then one of the following holds for $i \in [2]$.

1. Γ_i is connected in red and non-bipartite;
2. Γ_i is connected and bipartite. Furthermore, it has minimum degree at least $(1/4 - 39\eta)m$;
3. Γ_i consists of two red components, each of order at least $(1/4 - 39\eta)m$.

By the definition of a reduced graph and our choice of parameters, the number of blue edges of G which are not present in G' is at most $14\eta m^2$ and similarly for the red edges. It follows from Lemma 16 that one of the conditions in 3 holds. Suppose that Condition 2 holds for both Γ_1 and Γ_2 . Denote the bipartition of Γ_i by $\{Y_{i,1}, Y_{i,2}\}$. If there are no edges between $Y_{1,j}$ and $Y_{2,j}$ for $j = 1, 2$, Condition 3 is satisfied. Otherwise, without loss of generality, there is an edge between $Y_{1,1}$ and $Y_{2,1}$. Then the red graph Γ_B is connected. But we assumed that there Γ has no spanning non-bipartite monochromatic component, so there are no edges between $Y_{i,1}$ and $Y_{3-i,2}$ for $i \in [2]$. Thus, up to relabelling of $Y_{i,j}$, Condition 3 holds.

We assume now that Φ_1 does not span Γ . Denote $s = |\Phi_1|$. Suppose first that $s > (1/2 + 26\eta)m$. Denote $U = V(\Gamma) \setminus V(\Phi_1)$. Then every vertex $u \in U$, is incident to at least $(3/4 - 13\eta)m - (m - s) > s/2$ neighbours. It follows that every two vertices outside of $V(\Phi_1)$ have a common red neighbour, implying that U is contained in a red component Φ_2 of order at least $(3/4 - 13\eta)m$. Indeed, pick $u \in U$. Then the red neighbourhood of u in Γ is contained in Φ_2 as well as U .

By the choice of Φ_1 as the largest connected monochromatic subgraph, we have that $|\Phi_1| \geq (3/4 - 13\eta)m$. The components Φ_1, Φ_2 cover Γ and intersect in at least $(1/2 - 26\eta)n$ vertices. By the following claim, at least one of Φ_1, Φ_2 is non-bipartite, implying that Condition 4 holds, by Lemma 16.

Claim 32. *At least one of the graphs Φ_1, Φ_2 is not bipartite.*

Proof. Suppose otherwise. Denote $X_i = V(\Phi_i) \setminus V(\Phi_{3-i})$ for $i \in [2]$ and $Y = V(\Phi_1) \cap V(\Phi_2)$. Then for every $x_1 \in X_1, x_2 \in X_2$, the vertices x_1, x_2 are non-adjacent in Γ . Thus, x_1 sends at least $(3/4 - 13\eta)m - |X_1|$ (blue) edges to Y and similarly x_2 sends at least $(3/4 - 13\eta)m - |X_2|$ red edges into Y . If both Φ_1 and Φ_2 are bipartite, it follows that Y contains a set A_1 of $(3/4 - 13\eta)m - |X_1|$ vertices spanning no blue edges and a set A_2 of $(3/4 - 13\eta)m - |X_2|$ spanning no red edges. It follows that Y contains an independent set of size at least $|A_1| + |A_2| - |Y| \geq (3/2 - 26\eta)m - (|X_1| + |X_2| + |Y|) \geq (1/2 - 26\eta)m$. This is a contradiction to the minimum degree condition on Γ . \square

It remains to consider the case where $s = |\Phi_1| \leq (1/2 + 26\eta)m$. An argument similar to a previous one shows that if $s < (1/2 - 26\eta)m$, every two vertices of Φ_1 have a common red neighbour outside of Φ_1 , contradicting the choice of Φ_1 as the largest monochromatic component. Thus we have that $(1/2 - 26\eta)m \leq s \leq (1/2 + 26\eta)m$. Note that we may find $u_1, u_2 \in V(\Phi_1)$ which have no common red neighbour outside of Φ_1 (otherwise there is a red component of order larger than $|\Phi_1|$ contradicting our choice of Φ_1). Denote by X_i the set of red neighbours of u_i outside of $V(\Phi_1)$. Let Y_i be the red neighbourhood of X_i in Φ_1 . It follows from the minimum degree condition and the order of Φ_1 that $|X_i|, |Y_i| \geq (1/4 - 39\eta)m$. Furthermore, the sets X_1, X_2, Y_1, Y_2 are disjoint, there are no red edges in between $X_1 \cup Y_1$ and $X_2 \cup Y_2$ and no blue edges between $X_1 \cup X_2$ and $Y_1 \cup Y_2$. In particular, there are no edges between X_i and Y_{3-i} . Considering the minimum degree conditions and the size of the various sets, it follows that the blue subgraph $\Phi_2 = \Gamma_B \setminus V(\Phi_1)$ is connected. Similarly, $\Gamma_R[X_i \cup Y_i]$ is connected. Moreover, of the four components, there cannot be both a red and a blue bipartite component. Condition 5 follows. \square

Proof of the main theorem

We now prove Theorem 1, using Lemma 31 and other results which we shall state and prove in subsequent sections.

Theorem 1. *There exists n_0 such that if a graph on $n \geq n_0$ vertices and minimum degree at least $3n/4$ is 2-coloured then its vertex set may be partitioned into two monochromatic cycles of different colours.*

Proof of Theorem 1. Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on $n \geq n_0$ vertices with minimum degree at least $3n/4$ and a red and blue colouring of the edges. By Lemma 31, we may assume that one of Conditions 1 to 5 from the statement of the lemma hold. It remains to conclude that in each of these cases, we may find a partition of $V(G)$ into a red cycle and a blue one. We prove this for each of the above cases using lemmas appearing in Sections 8 to 13.

We start by resolving the easiest conditions, namely Condition 2, in Lemma 33 in Section 8. Condition 3 is dealt with in Sections 9, 10. Lemmas 35, 36 and 37 share some similarities and are used to prove all possible combinations of Conditions 3a, 3b, 3c except for the case where Condition 3b holds for both graphs in question. The proof of Theorem 1 in the latter case can be deduced from Lemma 38 and is of different nature. Condition 4 can be resolved by Lemma 41 in Section 11. It shares some ideas with the proofs in Section 9, but requires further analysis. Condition 5 is dealt with by Lemma 44 in Section 12. Condition 1 turns out to be hardest, thus we prove it last in Lemma 47 in Section 13. \square

8 Almost spanning weakly robust blue, large strongly robust red

In this section we resolve Condition 2 of Lemma 31. We recommend the reader to follow the proof here carefully, since the methods appearing here will be used in later sections, often in less detail.

In order to prove Theorem 1 in Condition 2, we prove the following lemma. In this case we make the further assumption that the given robust subgraphs cover all the vertices of G (rather than almost all of them). This can be easily justified by the fact that every vertex sends many edges to the intersection of the red and blue robust graphs (denoted below as F_1 and F_2), due to the size of the robust graphs and the minimum degree conditions, and by Lemma 5, which states that given a robust subgraph F , the graph obtained by adding vertices of large degree into F remains robust.

Lemma 33. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2 satisfy the following conditions.*

- $F_1 \subseteq G_B$ is (α, k) -weakly-robust with bipartition $\{X, Y\}$ and $|F_1| \geq (1 - \varepsilon)n$;
- $F_2 \subseteq G_R$ is (α, k) -strongly-robust and $|F_2| \geq (1/2 + 2\varepsilon)n$;
- $V(G) = V(F_1) \cup V(F_2)$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

This case is the most straightforward of the cases arising from Lemma 31. We apply Lemma 7 to find vertex-disjoint absorbing paths P_i in F_i for $i \in [2]$. Then, using the regularity lemma and the connected matching method, we find two vertex-disjoint monochromatic cycles containing the paths P_1, P_2 and covering almost all of the vertices. Finally, we use the absorption property to insert the remaining vertices into the paths P_1, P_2 so as to obtain the desired monochromatic cycle partition.

Proof of Lemma 33. We use Lemma 7 to build absorbing paths for F_1 and F_2 . More precisely, given a suitably small $\rho > 0$, there exists vertex-disjoint paths $P_i \subseteq F_i$ for $i \in [2]$ satisfying the following conditions, where $U = V(P_1) \cup V(P_2)$.

- $|P_i| \leq \rho n$;
- P_1 is $\rho^2 n$ -weakly-absorbing in F_1 (with respect to the bipartition $\{X, Y\}$);
- P_2 is $\rho^2 n$ -absorbing in F_2 .

Indeed, by Lemma 7, we may find a $\rho^2 n$ -weakly-absorbing path P_1 in F_1 of length at most ρn . By Lemma 3, the subgraph $F'_2 = F_2 \setminus V(P_1)$ is robust (with suitable parameters), thus we may find a $\rho^2 n$ -absorbing path P_2 in F'_2 of length at most ρn .

Using the regularity lemma, we shall find vertex-disjoint monochromatic cycles, containing the paths P_1 and P_2 and covering almost all vertices of G . We then cover the remaining vertices using

the absorption properties of the paths P_1 and P_2 . However, the fact that P_1 is weakly-absorbing presents a technical difficulty which we overcome as follows.

Recall that $\{X, Y\}$ is the bipartition of F_1 . Without loss of generality, suppose that $|X \cap V(F_2)| \geq n/8$. Note that $|Y| \geq \alpha n$ by the minimum degree condition on $F_1[X, Y]$. Pick subsets $A_1 \subseteq X \cap V(F_2)$ and $A_2 \subseteq Y$, disjoint of U , such that $|A_1| = \rho^2 n/2$ and $|A_2| = \rho^2 n/4$, and denote $A = A_1 \cup A_2$.

Apply Lemma 10 to the graph $G \setminus (U \cup A)$ with parameter $\eta = \rho^2/16$ and the cover $\{V(F_1), V(F_2)\}$. Let Γ be the corresponding $(\eta, 6\eta)$ -reduced graph. Note that $\delta(G \setminus (U \cup A)) \geq (3/4 - 3\rho)n$. It follows from observation 11 that for a suitable choice of ρ we have $\delta(\Gamma) \geq (3/4 - \varepsilon/2)m$, where $m = |\Gamma|$.

The blue subgraph of Γ determined by the clusters contained in $V(F_1)$ is connected, by Lemma 17. Let Φ_1 be the connected component of Γ_B containing that subgraph. Similarly, let Φ_2 be the connected component of Γ_R containing the clusters which are contained in $V(F_2)$. Note that Φ_1 and Φ_2 cover $V(\Gamma)$. Furthermore, $|\Phi_1| \geq (1 - 3\varepsilon/2)m$ and $|\Phi_2| \geq (1/2 + 3\varepsilon/2)m$. Let Γ' be the union of these graphs.

Claim 34. Γ' has a perfect matching.

Proof. Denote $V_1 = V(\Phi_1) \cap V(\Phi_2)$ and $V_2 = V(\Phi_1) \Delta V(\Phi_2)$. Note that V_1, V_2 partition $V(\Gamma)$ and $|V_1| \geq m/2$. Recall that $\delta(\Gamma) \geq (3/4 - \varepsilon/2)m$. It follows that for every $v \in V_1$, we have $\deg_{\Gamma'}(v) \geq (3/4 - \varepsilon/2)m$, because all edges of Γ incident to v are in Γ' . Vertices not in Φ_1 have blue degree at most $\frac{3}{2}\varepsilon m$. Similarly, vertices not in Φ_2 have red degree at most $(1/2 - 3\varepsilon/2)m$. It follows that for every $v \in V_2$, we have $\deg_{\Gamma'}(v) \geq (1/4 + \varepsilon)m$. We conclude from Theorem 19 that Γ has a Hamilton cycle and in particular a perfect matching. \square

Remark. In the last claim, we implicitly assumed that m is even. It is indeed possible to make this further assumption in the regularity lemma. We shall make this assumption whenever convenient without stating so explicitly.

By Claim 34, Γ has a perfect matching consisting of a blue connected matching in Φ_1 and a red connecting matching in Φ_2 . Thus, we may use Lemma 18 to obtain a blue cycle C_1 and a red cycle C_2 which are disjoint, each C_i contains the respective absorbing path P_i and together they cover all but at most $7\eta n \leq \rho^2 n/4$ vertices of $V(G) \setminus A$.

We now show how to absorb the leftover vertices into the cycles C_1, C_2 . Let B be the set of vertices which are not contained in the cycles C_1, C_2 or in the set A , so $|B| \leq \rho^2 n/4$. Denote $B_1 = X \cap B$, $B_2 = Y \cap B$, and $B_3 = B \setminus V(F_1)$. Recall that $A_1 \subseteq X \cap V(F_2)$ and $A_2 \subseteq Y$ are disjoint of the cycles C_1, C_2 and have sizes $\rho^2 n/2$ and $\rho^2 n/4$ respectively. It follows that $\rho^2 n/4 \leq |A_2 \cup B_2| \leq \rho^2 n/2$. Thus we may choose $A'_1 \subseteq A_1$ such that $|A'_1 \cup B_1| = |A_2 \cup B_2| \leq \rho^2 n/2$. Recall that the path P_1 , which is contained in C_1 , is $\rho^2 n$ -absorbing in F_1 , so the vertices in these sets can be absorbed by P_1 and thus by C_1 . We remain with the vertices $(A_1 \setminus A'_1) \cup B_3$. There are at most $\rho^2 n$ of them and they belong to F_2 , so we may absorb them into P_2 and thus into C_2 . This completes a partition of $V(G)$ into a red and a blue cycle. \square

Now that we have proved Theorem 1 for Condition 2 of Lemma 31, we are ready to consider harder cases.

9 Almost spanning, almost balanced, weakly robust blue

In this section we consider Condition 3 of Lemma 31, where we have a large weakly robust blue subgraph with an almost balanced bipartition $\{X_1, X_2\}$. There are three possibilities for each of the red graphs $G_R[X_i]$. In this section we focus on the case where at least one of these red subgraphs satisfies Conditions 3a or 3c. The remaining case, with both red graphs $G_R[X_1], G_R[X_2]$ satisfying Condition 3b requires a proof of different nature and is thus postponed to the following section.

The main idea in the various cases arising here is that if two red robust subgraphs may be joined by two vertex-disjoint red paths, then they can essentially be treated as one bigger component, in which case we may use the argument of the previous section to finish the proof. If that is not possible, we deduce that the red graph G_R may be disconnected by removing a small number of vertices into two almost half-sized subgraphs. This case may be resolved by Lemmas 29 and 30.

The following lemma resolves the case where one of the red graphs in question satisfies Condition 3a and the other satisfies 3a or 3b.

Lemma 35. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3 satisfy the following assertions.*

- $F_1 \subseteq G_B$ is (α, k) -weakly-robust and with bipartition $\{X, Y\}$, where $|X|, |Y| \geq (1/2 - \varepsilon)n$ and $e(G_B[X]), e(G_B[Y]) \leq \varepsilon n^2$;
- $F_2 \subseteq G_R$ is (α, k) -strongly-robust and $V(F_2) = X$;
- $F_3 \subseteq G_R$ is (α, k) -robust with $V(F_3) = Y$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

To prove this lemma, we show that we may either join the two robust subgraphs F_2, F_3 and then continue similarly to the proof of Lemma 33, or we can finish using Lemma 29.

Proof of Lemma 35. By Menger's theorem, one of the following holds.

1. There are two vertex-disjoint red paths P_2, P_3 , each with one end in F_2 and the other in F_3 ;
2. There exists $u \in V(G)$ such that $F_2 \setminus \{u\}$ and $F_3 \setminus \{u\}$ are disconnected from each other in $G_R \setminus \{u\}$.

In Case 2, it is easy to deduce that the required monochromatic cycle partition exists from Lemma 30. It remains to consider Case 1. The idea is to use the paths P_2 and P_3 so as to essentially connect the two subgraphs F_2, F_3 into one large component. We achieve this as follows.

Note that we may assume that the internal vertices of P_2, P_3 belong to $V(G) \setminus (V(F_2) \cup V(F_3))$. It follows that $|P_2|, |P_3| \leq 2\varepsilon n$ and the subgraphs F_1, F_2, F_3 remain robust after removing the vertices of the paths P_2, P_3 . Note that every vertex has either large (say, at least αn) blue degree into F_1 , or large red degree into either F_2 or F_3 . It follows that we may extend these subgraphs to cover the remaining vertices. Namely, using Lemma 5, we obtain $(\alpha/2, k+2)$ -robust subgraphs F'_1, F'_2, F'_3 which extend the given subgraphs and cover $V(G) \setminus (V(P_2) \cup V(P_3))$. Denote by $\{X', Y'\}$ the bipartition of F'_1 .

By Lemma 7, F'_i contains a $\rho^2 n$ -absorbing path P'_i of length at most ρn for each $i \in [3]$. We may assume that the paths P'_1, P'_2, P'_3 are vertex disjoint. For $i \in \{2, 3\}$, we may connect P_i with P'_i in F_i to a path Q_i using at most $k+2$ additional vertices. For convenience, we denote $Q_1 = P'_1$. To conclude, the paths $Q_1 \subseteq G_B$ and $Q_2, Q_3 \subseteq G_R$ are vertex-disjoint paths, such that Q_i is $\rho^2 n$ -absorbing in F'_i . Furthermore, each of Q_2, Q_3 has one end in F'_2 and one in F'_3 .

Recall that F_1 is weakly robust and that F_2 is strongly robust. It is perhaps not evident from the definitions that a strongly robust graph is weakly robust, but for our purpose, the place where the difference is important is in the definition of an absorbing path. But clearly, an absorbing path is weakly absorbing. Thus, it suffices to consider the case where F'_3 is weakly robust with bipartition $\{Z_1, Z_2\}$.

Similarly to the proof in Section 8, in order to overcome the technical issues arising when dealing with weakly robust subgraphs, we pick sets A_1, A_2, A_3 such that the following holds.

- A_1, A_2, A_3 are pairwise vertex-disjoint and do not intersect $V(Q_1) \cup V(Q_2) \cup V(Q_3)$;
- $A_1 \subseteq Z_1 \cap Y$, $A_2 \subseteq Z_2 \cap Y$ and $A_3 \subseteq X \cap V(F_2)$;
- $|A_1| = \rho^2 n/16$, $|A_2| = \rho^2 n/4$ and $|A_3| = \rho^2 n/2$.

We now consider the $(\eta, 6\eta)$ -reduced graph obtaining by applying Lemma 10 to the graph obtained from G by removing the vertices in the Q_i 's and A_i 's (with the cover $\{V(F'_1), V(F'_2), V(F'_3)\}$). The reduced graph Γ consists of a large blue component Φ_1 (containing almost all vertices) and two disjoint almost half-sized connected red subgraphs Φ_2, Φ_3 . It is easy to verify, similarly to the proof of Claim 34, using Theorem 19, that Γ has a perfect matching consisting of edges in Φ_1, Φ_2 and Φ_3 .

By Lemma 18, there exist a blue cycle C_1 and a red cycle C_2 with the following properties.

- C_1 and C_2 are vertex-disjoint and do not intersect $A = A_1 \cup A_2 \cup A_3$;
- They cover all but at most $4\eta n$ vertices of $V(G) \setminus A$;
- C_1 contains the path Q_1 and C_2 contains the paths Q_2, Q_3 .

Let us elaborate slightly more on how to obtain the required cycles by pointing out that C_2 may be obtained by connecting the ends of Q_2, Q_3 by two paths, one in F'_2 and the other in F'_3 . Let B be the set of vertices which do not belong to the cycles C_1, C_2 or to A (so $|B| \leq 7\eta n \leq \rho^2 n/16$). We define sets B_i as follows.

$$\begin{aligned} B_1 &= B \cap Z_1, & B_2 &= B \cap Z_2, \\ B_3 &= (B \cap X') \setminus (B_1 \cup B_2), \\ B_4 &= (B \cap Y') \setminus (B_1 \cup B_2), \\ B_5 &= B \setminus (B_1 \cup \dots \cup B_4). \end{aligned}$$

We perform the following steps in order to absorb B .

- We have $|B_2| \leq \rho^2 n/16 \leq |A_1 \cup B_1| \leq \rho^2 n/8$ and $|A_2| = \rho^2 n/4$. Thus we may choose $A'_2 \subseteq A_2$ such that $|A_1 \cup B_1| = |A'_2 \cup B_2|$. The vertices $A_1 \cup B_1 \cup A'_2 \cup B_2$ can be absorbed into Q_3 .
- Similarly, we may choose $A'_3 \subseteq A_3$ such that $|(A_2 \setminus A'_2) \cup B_3| = |A'_3 \cup B_4|$. The vertices $(A_2 \setminus A'_2) \cup B_3 \cup A'_3 \cup B_4$ can be absorbed into Q_1 .
- The remaining vertices $(A_3 \setminus A'_3) \cup B_5$ can be absorbed into Q_2 .

This completes the partition of $V(G)$ into a blue cycle and a red one. □

The following lemma resolves the case where one of the red graphs in question satisfies Condition 3c in Lemma 31 and the other satisfies one of the other two conditions.

Lemma 36. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3, F_4 satisfy the following assertions.*

- $F_1 \subseteq G_B$ is (α, k) -weakly-robust and with bipartition $\{X, Y\}$, where $|X|, |Y| \geq (1/2 - \varepsilon)n$;
- $F_2 \subseteq G_R$ is (α, k) -robust and $V(F_2) = X$;
- $F_3, F_4 \subseteq G_R$ are (α, k) -strongly-robust subgraphs of order at least $(1/4 - \varepsilon)n$ whose vertex sets partition Y . Furthermore, $e(G_R[V(F_3), V(F_4)]) \leq \varepsilon n^2$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

The proof of this lemma is similar to the previous one. Either F_2 may be joined to one of F_3, F_4 or we may finish using Lemma 29.

Proof of Lemma 36. Similarly to the previous case, two possibilities arise.

1. There exist vertex-disjoint red paths P_1, P_2 with one end in F_2 and either both have the other end in F_3 or both have the other end in F_4 ;

2. There exists a set U of size at most 2 such that $F_1 \setminus U$ and $(F_2 \cup F_3) \setminus U$ are disconnected in $G_R \setminus U$.

In Case 2, the required monochromatic cycle partition exists by Lemma 29. Indeed, we may find $S \subseteq X \setminus U$ and $T \subseteq Y \setminus U$ such that the three conditions in the lemma hold, for some parameter $\eta = \eta(\varepsilon)$. In particular, the third condition holds because most vertices in Y have degree at most $(1/4 + \sqrt{\varepsilon})n$ in $G[Y]$, and thus have degree at least $(1/2 - \sqrt{\varepsilon})n$ into X . If Case 1 holds, we may continue as in the proof of Lemma 35. \square

We now consider the last remaining case in this section, where both red graphs satisfy Condition 3c in Lemma 31.

Lemma 37. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3, F_4, F_5 satisfy the following conditions.*

- $F_1 \subseteq G_B$ is (α, k) -weakly-robust and with bipartition $\{X, Y\}$, where $|X|, |Y| \geq (1/2 - \varepsilon)n$, and $e(G_B[X]), e(G_B[Y]) \leq \varepsilon n^2$;
- $F_2, F_3 \subseteq G_B$ are (α, k) -strongly-robust subgraphs of order at least $(1/4 - \varepsilon)n$ whose vertex sets partition X , and $e(G_R[V(F_2), V(F_3)]) \leq \varepsilon n^2$;
- $F_4, F_5 \subseteq G_B$ are (α, k) -strongly-robust subgraphs of order at least $(1/4 - \varepsilon)n$ whose vertex sets partition Y , and $e(G_R[V(F_4), V(F_5)]) \leq \varepsilon n^2$;
- $e(G_R[V(F_2), V(F_3)]), e(G_R[V(F_4), V(F_5)]) \leq \varepsilon n^2$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

To prove this lemma we follow similar ideas to the previous results in this section. We show that either at least three of the four components F_2, F_3, F_4, F_5 may be joined or we may finish using Lemma 29 or Lemma 30.

Proof of Lemma 37. We consider four cases. In order to be able to distinguish between them, we define a graph H on vertex set $\{2, 3, 4, 5\}$ with an edge (i, j) , where $i \in \{2, 3\}, j \in \{4, 5\}$ if $e(G_R[F_i, F_j]) \geq \varepsilon n^2$ (so H is a bipartite graph on four vertices with bipartition $\{\{2, 3\}, \{4, 5\}\}$). Clearly, one of the following conditions.

1. H contains a path of length 2;
2. H consist of two vertex-disjoint edges;
3. H has exactly one edge;
4. H has no edges.

In Case 1, without loss of generality, $(2, 4), (2, 5) \in E(H)$, so $e(G_R[F_2, F_4]), e(G_R[F_2, F_5]) \geq \varepsilon n^2$. In particular, there exist four vertex-disjoint edges $e_1, e_2 \in G_R[F_2, F_4], e_3, e_4 \in G_R[F_2, F_5]$. We deduce that the components F_2, F_4, F_5 may be joined to form a red component of size at least $(3/4 - 3\varepsilon)n$. We may now finish the proof of Lemma 37 similarly to the previous lemmas in this section.

Suppose that Case 2 holds. Without loss of generality, $E(H) = \{(2, 4), (3, 5)\}$. If there are two vertex-disjoint red paths P_1, P_2 between $V(F_2) \cup V(F_4)$ and $V(F_3) \cup V(F_5)$, we conclude that the four component F_2, F_3, F_4, F_5 may be joined. Otherwise, there is a vertex $u \in V(G)$ such that $G_R \setminus \{u\}$ is disconnected, with F_2, F_4 in one component and F_3, F_5 in another. The proof of Lemma 37 can be completed by Lemma 30.

We now consider Case 3. Without loss of generality, $e(G_R[V(F_2), V(F_4)]) \geq \varepsilon n^2$. Note that in this case, almost all edges between F_2 and F_5 , F_5 and F_3 , F_3 and F_4 are blue. Additionally, almost all edges within the F_i 's are red. Thus the proof follows from Lemma 29, unless almost all edges between F_2 and F_4 are red. In the latter case, we may find the required cycle partition by hand, as can be seen by the following sketch. Note that all but $c\varepsilon n$ (for a suitable constant c) vertices in F_2 have red degree into F_2 , red degree into F_4 and blue degree into F_5 at least $3n/16$. Let A_2 be the set of such vertices. A similar statement holds for F_3, F_4, F_5 , we denote by A_3, A_4, A_5 the corresponding subsets. We note that each of the remaining vertices either red or blue degree at least $n/16$ into $A_2 \cup A_4$. It is easy to see that there exist a red path P_1 with ends in $A_2 \cup A_4$ and a blue path P_2 with end in $A_3 \cup A_5$ whose length is at most $c'\varepsilon n$ that cover all the vertices not in $A_2 \cup A_3 \cup A_4 \cup A_5$. In fact, we may assume that the number of vertices remaining in A_3 is equal to the number of vertices remaining in A_5 (by using blue edges between A_3 and A_4 or between A_5 and A_2). It easily follows that P_2 may be extended to a cycle C_2 that covers $A_3 \cup A_5$. Using the high minimum red degree in $A_2 \cup A_4$, we may extend P_1 to a red cycle C_1 whose vertex set is $V(G) \setminus V(C_2)$, thus we obtain the required cycle partition.

If Case 4 holds, the proof may be completed by 29, as almost all edges between X and Y are blue. \square

In order to finish the proof of Theorem 1 under the assumption that Condition 3 from Lemma 31, we need to consider the case where both graphs in question satisfy Condition 3b. This is done in the next section, Section 10.

10 Almost spanning weakly robust blue, two half-sized weakly robust red

In this section we consider Condition 3 from Lemma 31, where both graphs $G_R[X_1]$ and $G_R[X_2]$ satisfy Condition 3b. Namely, we have a large weakly robust blue subgraph with an almost balanced bipartition $\{X_1, X_2\}$. Furthermore, $G_R[X_i]$ is weakly robust for $i \in [2]$ with an almost balanced bipartition $\{Y_{i,1}, Y_{i,2}\}$ such that $e(G_R[Y_{i,1}, Y_{i,2}]) \leq \varepsilon n^2$.

Lemma 38. Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that there exists four disjoint sets $Y_{i,j}$, $i, j \in [2]$ with the following properties.

- $|Y_{i,j}| \geq (1/4 - \varepsilon)n$ for $i, j \in [2]$;
- $e(G[Y_{i,j}]) \leq \varepsilon n^2$;
- $e(G_B[Y_{i,1}, Y_{i,2}]) \leq \varepsilon n^2$ for $i \in [2]$;
- $e(G_R[Y_{1,j}, Y_{2,j}]) \leq \varepsilon n^2$ for $j \in [2]$.

We notice that a graph with the given conditions has a rather specific structure. Namely, the sets $Y_{i,j}$ span few edges, whereas the graphs $G_R[Y_{i,1}, Y_{i,2}]$ and $G_B[Y_{1,j}, Y_{2,j}]$ are almost complete. By Lemma 29, we conclude that we may finish the proof unless say $G_R[Y_{1,1}, Y_{2,2}]$ and $G_B[Y_{1,2}, Y_{2,1}]$ are almost complete. In the latter case we construct the required partition into a red cycle and a blue one ‘by hand’.

Proof of Lemma 38. The conditions imply that for some $\eta = \eta(\varepsilon)$, we can find disjoint sets S_1, S_2, T_1, T_2 with the following properties.

- $|S_i|, |T_i| \geq (1/4 - \eta)n$ for $i \in [2]$;
- $\delta(G_R[S_1, S_2]) \geq (1/4 - \eta)n$ and $\delta(G_R[T_1, T_2]) \geq (1/4 - \eta)n$;
- $\delta(G_B[S_i, T_i]) \geq (1/4 - \eta)n$ for $i \in [2]$;
- $e(G[S_i]), e(G[T_i]) \leq \eta n^2$.

Denote $S = S_1 \cup S_2$ and $T = T_1 \cup T_2$. Consider the graph $G_B[S, T]$. If for every $S' \subseteq S$ and $T' \subseteq T$ with $|S'|, |T'| \geq (1/4 - 200\eta)n$ we have $e(G_B[S', T']) \geq 50\eta n^2$, Lemma 29 implies that G may be partitioned into a blue cycle and a red one. Thus we may assume that there exist subsets $S' \subseteq S$ and $T' \subseteq T$ of size $(1/4 - 200\eta)n$ such that $e(G_B[S', T']) \leq 50\eta n^2$.

Claim 39. Either $|S_1 \cap S'|, |T_2 \cap T'| \leq 10\sqrt{\eta}n$ or $|S_2 \cap S'|, |T_1 \cap T'| \leq 10\sqrt{\eta}n$.

Proof. Suppose to the contrary that $|S_1 \cap S'|, |T_1 \cap T'| \geq 10\sqrt{\eta}n$. For every $u \in S_1$, the number of vertices in T_1 which are not blue neighbours of u is at most $4\eta n$. If $S' \cap S_1$ and $T' \cap T_1$ both have size at least $10\sqrt{\eta}n$, we deduce $e(G_B[S', T']) \geq 90\eta n^2$, a contradiction. \square

By the above claim, without loss of generality, $|S_2 \cap S'|, |T_1 \cap T'| \leq 10\sqrt{\eta}n$, so $G_B[S_1, T_2]$ is almost empty, and $G_R[S_1, T_2]$ is almost complete. Similarly, $G_B[S_2, T_1]$ is almost complete. We deduce that for some $\rho = \rho(\varepsilon)$ there exist disjoint sets A_1, A_2, A_3, A_4 satisfying the following conditions.

- $|A_1|, |A_2|, |A_3|, |A_4| \geq (1/4 - \rho)n$;
- $\delta(G_B[A_1, A_2]), \delta(G_B[A_2, A_3]), \delta(G_B[A_3, A_4]) \geq (1/4 - \rho)n$;
- $\delta(G_R[A_1, A_3]), \delta(G_R[A_1, A_4]), \delta(G_R[A_2, A_4]) \geq (1/4 - \rho)n$.

We obtain a partition $\{A'_1, A'_2, A'_3, A'_4\}$ of $V(G)$ by adding each vertex $u \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$, to one of the sets A_1, A_2, A_3, A_4 as follows.

$$\text{If } \begin{array}{l} \deg_B(u, A_2) \geq n/32 \\ \deg_B(u, A_3) \geq n/32 \\ \deg_R(u, A_2) \geq n/32 \\ \deg_R(u, A_3) \geq n/32 \end{array} \quad \text{add } u \text{ to } \begin{array}{l} A_3 \\ A_2 \\ A_4 \\ A_1. \end{array}$$

Note that every vertex will be added to one of the A_i 's. We first consider the case where n is even; the case where n is odd is similar but there is small added technicality.

Case 1: n is even

Denote $m_i = |A'_i|$. We will find values k_1, k_2, k_3 and l_1, l_2, l_3 and a partition of G into a blue cycle C_1 and a red cycle C_2 such that the following holds.

$$|V(C_1) \cap A'_i| = \begin{cases} k_1 & i = 1 \\ k_1 + k_2 & i = 2 \\ k_2 + k_3 & i = 3 \\ k_3 & i = 4. \end{cases} \quad |V(C_2) \cap A'_i| = \begin{cases} l_1 + l_2 & i = 1 \\ l_3 & i = 2 \\ l_1 & i = 3 \\ l_2 + l_3 & i = 4. \end{cases}$$

More precisely, the blue cycle C_1 will be comprised of paths P_1, P_2, P_3, P_4 (so $C_1 = P_1 P_2 P_3 P_4$) with the following properties.

- $P_1 \in G_B[A'_1, A'_2]$, its ends are in A'_2 and it has k_1 vertices in A'_1 ;
- $P_2, P_4 \in G_B[A'_2, A'_3]$, both have one end in A'_2 and the other in A'_3 and together they have $k_2 + 1$ vertices in A'_2 ;
- $P_3 \in G_B[A'_3, A'_4]$, its ends are in A'_3 and it has k_3 vertices in A'_4 .

Similarly, we find red paths Q_1, Q_2, Q_3, Q_4 forming a red cycle $C_2 = Q_1 Q_2 Q_3 Q_4$ with the following properties.

- $Q_1 \in G_R[A'_1, A'_3]$, its ends are in A'_1 and it has l_1 vertices in A'_3 ;
- $Q_2, Q_3 \in G_R[A'_1, A'_4]$, both have one end in A'_1 and the other in A'_4 and together they have $l_2 + 1$ vertices in A'_1 ;

- $Q_4 \in G_R[A'_2, A'_4]$, its ends are in A'_2 and it has l_3 vertices in A'_2 .

The values k_1, k_2, k_3 and l_1, l_2, l_3 clearly need to satisfy the following system of equations.

$$\begin{aligned} m_1 &= k_1 + l_1 + l_2 \\ m_2 &= k_1 + k_2 + l_3 \\ m_3 &= k_2 + k_3 + l_1 \\ m_4 &= k_3 + l_2 + l_3. \end{aligned}$$

Which may be solved as follows.

$$\begin{aligned} k_1 &= k_3 + \frac{1}{2}(m_1 + m_2 - m_3 - m_4) \\ l_1 &= l_3 + \frac{1}{2}(m_1 - m_2 + m_3 - m_4) \\ k_2 &= -(k_3 + l_3) + m_4 \\ l_2 &= -(k_3 + l_3) + m_3 - \frac{1}{2}(m_1 - m_2 + m_3 - m_4). \end{aligned}$$

Since n is even, if k_3, l_3 are integers then so are k_1, k_2, l_1, l_2 . Note that the following inequalities hold.

$$\begin{aligned} k_3 - 2\rho n &\leq k_1 \leq k_3 + 2\rho n & k_2 &\geq (1/4 - \rho)n - (k_3 + l_3) \\ l_3 - 2\rho n &\leq l_1 \leq l_3 + 2\rho n & l_2 &\geq (1/4 - 3\rho)n - (k_3 + l_3). \end{aligned}$$

We pick $l_3 = k_3 = \lceil 12\rho n \rceil$. It follows that $10\rho n \leq l_1, k_1 \leq 14\rho n + 1$.

Claim 40. *The required cycle partition exists.*

Proof. It is not hard to see that there exists a blue path P , that can be written as $P = P_1P_2P_3$ such that P has one end in A'_2 and one in A'_3 , and it has k_1 vertices in A'_1 , $k_1 + 1$ vertices in A'_2 , $k_2 + 1$ vertices in A'_3 and k_2 vertices in A'_4 .

Similarly, there is a red path Q whose ends are in A'_1 and A'_4 , and it has l_1 vertices in A'_3 , $l_1 + 1$ vertices in A'_1 , $l_2 + 1$ in A'_4 and l_2 in A'_2 . Furthermore, we may choose Q to be disjoint of P .

Let A''_i be the set of vertices in A'_i that are not in the interior of P or Q . Then $|A''_1| = m_1 - k_1 - l_1 = l_2$, and, similarly, $|A''_2| = |A''_3| = k_2$ and $|A''_4| = |A''_1| = l_2$. It follows from Corollary 20 that P may be completed into a blue cycle C_1 by a Hamilton cycle in $G_B[A''_2, A''_3]$ with the same ends as P . Q can be similarly completed into a red cycle C_2 , and the required cycle partition exists. \square

Case 2: n is odd

It remains to consider the case where n is odd. Without loss of generality, $|A'_1| > n/4$. By the minimum degree condition, this implies that there is an edge uv in A'_1 . We assume here that this

edge is blue; the proof in case the edge is red is very similar. We obtain the required cycle partition by picking a blue cycle $C_1 = P_1P_2P_3P_4$ and a red cycle $C_2 = Q_1Q_2Q_3Q_4$ where the P_i 's and Q_i 's are chosen as before, except for P_1 which is required to consist of edges in $G_B[A'_1, A'_2]$ and the edge uv , to have its ends in A'_2 , and have $k_1 + 1$ vertices in both A'_1 and A'_2 .

We obtain the same system of equations for k_i, l_i , except that m_1 is replaced by $m_1 - 1$. As n is odd, we find an integer solution with $l_3 = k_3 = \lceil 12\rho n \rceil$, $10\rho n - 1 \leq l_1, k_1 \leq 14\rho n + 2$. We proceed as before (it is easy to make sure that P_1 satisfies the new requirements). \square

The proof of Lemma 38 completes the proof of our main Theorem under the assumption that Condition 3 from Lemma 31 holds.

11 Red and blue robust subgraphs, size almost $3n/4$

In this section, we consider Condition 4 from Lemma 31. Similarly to Section 8, we may assume that the given robust subgraphs cover $V(G)$.

Lemma 41. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2 satisfy the following assertions.*

- $F_1 \subseteq G_B$ is (α, k) -strongly-robust and $|F_1| \geq (3/4 - \varepsilon)n$;
- $F_2 \subseteq G_R$ is (α, k) -robust and $|F_2| \geq (3/4 - \varepsilon)n$;
- $V(G) = V(F_1) \cup V(F_2)$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

We proceed as before, building absorbing paths, and considering the reduced graph on the remaining vertices, where we have a blue component and a red one, each with almost $3/4$ of the vertices. If a perfect matching can be found using the edges in these component, we continue as before to obtain the required partition into cycles. If no such perfect matching exists, we conclude that the graph G satisfies some structural conditions which enable us to either find the required partition ‘by hand’, or to join two components in a similar way to previous cases.

Proof of Lemma 41. Let Q_1, Q_2 be disjoint $\rho^2 n$ -absorbing paths in F_1, F_2 , respectively, of length at most ρn . As pointed out in Section 9 we may assume that F_2 is weakly robust. Denote its bipartition by $\{X, Y\}$.

Without loss of generality, $|X \cap V(F_1)| \geq n/8$ and $|Y| \geq \alpha n$. Fix sets $A_1 \subseteq X \cap V(F_1)$ of order $\rho^2 n/2$ and $A_2 \subseteq Y$ of order $\rho^2 n/4$. Finally, apply Lemma 10 (after removing the vertices in Q_1, Q_2, A_1, A_2) with small enough η . Consider the $(\eta, 6\eta)$ -reduced graph Γ . Note that, assuming ρ, η are small enough, we have $\delta(\Gamma) \geq (3/4 - \varepsilon)m$, where $m = |\Gamma|$. Furthermore, there is a blue

component Φ_1 and a red component Φ_2 which cover $V(\Gamma)$ and are of order at least $(3/4 - 2\varepsilon)m$ each. We consider the subgraph Γ' of Γ spanned by the blue edges in Φ_1 and the red ones in Φ_2 .

Claim 42. *One of the following conditions holds.*

1. Γ' has a perfect matching;
2. There exist subsets $V_1 \subseteq V(F_1) \setminus V(F_2)$, $V_2 \subseteq V(F_2) \setminus V(F_1)$ and $V_0 \subseteq V(F_1) \cap V(F_2)$ with the following properties (where η, β, l depend only on ε, α, k).
 - (a) $|V_1|, |V_2| \geq (1/4 - \eta)n$ and the graphs $G_R[V_1], G_B[V_2]$ are (β, l) -strongly-robust;
 - (b) $|V_0| \geq (1/2 - \eta)n$ and $G_B[V_1 \cup V_0]$ and $G_R[V_2 \cup V_0]$ are (β, l) -robust, with at least one of them being strongly-robust.

Proof. Let $U_1 = V(\Phi_1) \setminus V(\Phi_2)$, $U_2 = V(\Phi_2) \setminus V(\Phi_1)$ and $U_0 = V(\Phi_1 \cap \Phi_2)$. Note that the vertices in $U_1 \cup U_2$ have degree at least $(1/2 - 4\varepsilon)m$ in Γ' , whereas $|U_0| \geq (1/2 - 4\varepsilon)m$ and the vertices in U_3 have degree at least $(3/4 - 2\varepsilon)m$ in Γ' . If Γ' has no perfect matching, it follows from Theorem 19 that $|U_1 \cup U_2| \geq m/2$ and that the set W of vertices $U_1 \cup U_2$ with degree at most $(1/2 + \varepsilon)m$ has size at least $(1/2 - 4\varepsilon)m$. Denote $W_1 = W \cap U_1$ and $W_2 = W \cap U_2$.

We claim that $|W_1|, |W_2| \geq (1/4 - 13\varepsilon)n$. Indeed, the vertices in W_1 have blue degree at most $(1/2 + \varepsilon)m$, so they have red degree at least $(1/4 - 5\varepsilon)m$. Note that the red neighbourhood of W_1 is contained in U_1 . It follows that $|U_1| \geq (1/4 - 5\varepsilon)n$ and similarly $|U_2| \geq (1/4 - 5\varepsilon)n$. Since $|U_0| \geq (1/2 - 4\varepsilon)m$, we have that $|U_1|, |U_2| \leq (1/4 + 9\varepsilon)m$. Thus $|W_1|, |W_2| \geq (1/4 - 13\varepsilon)m$. It is easy to verify that $\Gamma_R[W_1]$ and $\Gamma_B[W_2]$ are connected and non bipartite (in fact, they are almost complete).

Let W_0 be the intersection of the blue neighbourhood of W_1 in U_0 and the red neighbourhood of W_2 in U_0 . By the definition of the sets U_i , there are no edges between U_1 and U_2 , there are no red edges between U_1 and U_0 and no blue edges between U_2 and U_0 . It follows that the each vertex in U_1 has at least $(1/2 - 10\varepsilon)m$ blue neighbours in U_0 , and by the analogous argument for U_2 , we have $|W_0| \geq (1/2 - 20\varepsilon)m$. Note that $\Gamma_B[W_0 \cup W_2]$ and $\Gamma_R[W_0 \cup W_1]$ are connected and it is not hard to see that at least one of them is non-bipartite. It follows from Lemma 16 that Condition 2 holds. \square

If Γ' has a perfect matching, we may find the required cycle partition as before (see, e.g. Section 8). We thus assume that the second condition holds. Similarly to the proofs in Section 9, one of the following holds.

1. There are two vertex-disjoint blue paths between V_2 and $V_1 \cup V_0$;
2. There are two vertex-disjoint red paths between V_1 and $V_2 \cup V_0$;
3. There exists a set X of size at most 2 such that the sets V_2 and $V_1 \cup V_0$ are not connected to each other in $G_B \setminus X$ and the sets V_1 and $V_2 \cup V_0$ are not connected to each other in $G_R \setminus X$.

In the first two cases we proceed as in Section 9 to join, say, the two blue component to form an almost spanning component, which together with the large red component $(V_2 \cup V_0)$ may be used to find the required cycle partition. Thus we assume that Case 3 holds.

Consider the graph $G' = G \setminus X$. Let U_1 be the component of V_1 in G'_R and let U_2 be the blue component of V_2 in G'_B . Then $e(G[U_1, U_2]) = 0$, so $|U_1|, |U_2| \leq n/4 - 1$.

Denote $W = V(G') \setminus (U_1 \cup U_2)$ (so $|W| \geq n/2$). We define U'_1, U'_2 as follows. For each $x \in X$, if $\deg_B(x, W) \geq n/8$, put x into U_1 . Otherwise we have $\deg_R(x, W) \geq n/8$, and we put x into U_2 . Note that $e(G[W]) \geq n^2/16$. We may assume without loss of generality that $e(G_B[W]) \geq n^2/32$. Denote $|W| = |U'_1| + |U'_2| + k$ and note that $0 \leq k \leq 5\eta n$.

Claim 43. *There exists $\theta = \theta(\varepsilon)$ such that W may be partitioned into sets W_1, W_2 satisfying the following conditions.*

- $|W_1| = |U'_1| + k$ and $|W_2| = |U'_2|$;
- $G_B[W_1]$ contains a blue path of length k ;
- The graphs $G_B[U'_1, W_1]$ and $G_R[U'_2, W_2]$ have minimum degree at least $n/32$ and all but at most θn vertices have degree at least $(1/4 - \theta)n$.

It is easy to conclude from Claim 43 that G may be partitioned into a blue cycle and a red one. Indeed, the graph $G_R[U'_2, W_2]$ is Hamiltonian (e.g. by Corollary 20). We claim that $G_B[U'_1 \cup W_1]$ is also Hamiltonian, completing the partition of $V(G)$ into a red cycle and a blue one. Indeed, take any path P in $G_B[W_1]$ of length k . Denote its ends by u, v and let W'_1 be the set obtained from W_1 by removing the interior vertices of P . It is easy to see (e.g. by Corollary 20) that $G_B[U'_1, W'_1]$ has a Hamilton path with ends u, v . It remains to prove Claim 43.

Proof of Claim 43. Recall that $(1/4 - \eta)n \leq |U_1|, |U_2| \leq n/4 - 1$. Denote by W' the set of vertices which have at least $\sqrt{\eta}n$ non-neighbours in either U_1 or U_2 . Note that each vertex in U_1 has at most ηn non-neighbours in W . It follows that $|W'| \leq \sqrt{\eta}n$.

Let W'_1 be the set of vertices in W' with (blue) degree at least $n/16$ into U'_1 , and let $W'_2 = W' \setminus W'_1$. So the vertices in W'_2 have degree at least $n/16$ into U'_2 . Note that $e(G_B[W \setminus W']) \geq n^2/64$. It follows from Theorem 21 that $W \setminus W'$ contains a blue path P of length k .

Pick any partition $\{W_1, W_2\}$ with the following properties.

- $|W_1| = |U'_1| + k$ and $|W_2| = |U'_2|$;
- $W'_1 \cup V(P) \subseteq W_1$ and $W'_2 \subseteq W_2$.

Such a partition satisfies the required conditions of Claim 43 with $\theta = \sqrt{\eta}$. □

□

12 Four half-sized robust subgraphs

In this section we consider Condition 5 from Lemma 31.

Lemma 44. *Let $\frac{1}{n} \ll \alpha, \frac{1}{k} \ll \varepsilon \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3, F_4 satisfy the following conditions.*

- $F_1, F_2 \subseteq G_B$ are vertex-disjoint (α, k) -robust subgraphs on at least $(1/2 - \varepsilon)n$ vertices;
- $F_3, F_4 \subseteq G_R$ are vertex-disjoint (α, k) -strongly-robust subgraphs on at least $(1/2 - \varepsilon)n$ vertices;
- $V(F_1) \cup V(F_2) = V(F_3) \cup V(F_4)$;
- $|V(F_i) \cap V(F_j)| \geq (1/4 - \varepsilon)n$ for $i \in [2], j \in [3, 4]$.

Then $V(G)$ may be partitioned into a blue cycle and a red one.

We follow similar ideas to previous sections. If there exist four vertex-disjoint paths, two of which are blue and connect F_1 with F_2 and two are red and connect F_3 with F_4 , then we may continue as in previous sections, by essentially having two robust subgraphs, one red and one blue, which both span almost all the vertices. The main effort in this case goes into showing that if such paths do not exist, the desired partition may be found by Lemma 30.

Proof of Lemma 44. We extend the components F_i as follows. For every vertex v not in any of the components, if there exist $i \in \{1, 2\}$ and $j \in \{3, 4\}$ such that v sends at least αn blue edges to F_i and at least αn red edges to F_j , we add v to F_i and F_j . Note that if no such i, j exist, then v either blue degree at least $(3/4 - 3\varepsilon)n$ or red degree at least $(3/4 - 3\varepsilon)n$.

Note that the obtained components satisfy the conditions above (though with relaxed parameters α, k in the definition of robustness and with say 2ε instead of ε). We abuse notation by denoting the modified components by F_1, F_2, F_3, F_4 . So in addition to the above conditions, we have that every vertex not in $V(F_1) \cup V(F_2)$ has either blue degree or red degree at least $(3/4 - 3\varepsilon)n$.

We claim that one of the following assertions holds.

1. There exist vertex-disjoint paths P_1, P_2, P_3, P_4 such that P_1, P_2 are blue paths from F_1 to F_2 and P_3, P_4 are red paths from F_3 to F_4 ;
2. There exist two vertices u, v such that F_1, F_2 belong to different connected components of $G_B \setminus \{u, v\}$. Furthermore, v sends at most εn blue edges to either F_1 or F_2 ;
3. There exist two vertices u, v such that F_3, F_4 belong to different connected components of $G_R \setminus \{u, v\}$. Furthermore, v sends at most εn red edges to either F_3 or F_4 .

Condition 1 implies that we may connect F_1 and F_2 using P_1, P_2 to obtain a large robust blue subgraph, and similarly we may connect F_3 and F_4 using the paths P_3, P_4 to obtain a large strongly robust red subgraph. We may continue as in Section 9 to obtain the desired partition of $V(G)$ into a red cycle and a blue one. If one of Conditions 2 and 3 holds, we may find the desired partition into a red cycle and a blue one by Lemma 30. It remains to prove that one of the above three conditions holds.

We call a vertex *blue* if it sends at least four blue edges to both F_1 and F_2 . Similarly, a vertex is *red* if it sends at least four red edges to both F_3 and F_4 . We show that either one of the above three conditions holds, or there are at least four vertices that are either blue or red. This would complete the proof of Lemma 44, due to the following claim.

Claim 45. *If there are four vertices that are either red or blue then one of Conditions 1 to 3 holds.*

Proof. It is easy to see that if there are two red vertices and two different vertices that are blue, then Condition 1 holds (take paths of length 2 through the red and blue vertices). Thus we may assume that there is at most one red vertex.

Suppose that Condition 3 does not hold. We claim that this implies that there are three vertex-disjoint red paths P_1, P_2, P_3 between F_3 and F_4 . Indeed, if no such paths exist, then there are two vertices u, v such that F_3, F_4 belong to distinct connected components of $G_R \setminus \{u, v\}$. Without loss of generality, v is not red, so it sends at most three red edges to one of F_3, F_4 , implying that Condition 3 holds, a contradiction.

From now on we assume that such paths do exist. We also assume that the interiors of P_1, P_2, P_3 do not intersect F_3 or F_4 . If there are four blue vertices u_1, u_2, u_3, u_4 , without loss of generality, $u_1, u_2 \notin V(P_1) \cup V(P_2)$ and we may find two vertex-disjoint blue paths between F_1 and F_2 which are disjoint of P_1, P_2 using the blue vertices u_1, u_2 (take them to be of length 2 and centred at u_1 and u_2). If there are three blue vertices u_1, u_2, u_3 and a different red vertex v , without loss of generality, the path P_1 has length 2, is centred at v and avoids u_1, u_2, u_3 . We may further assume that P_2 does not contain u_1, u_2 . It follows that there are two vertex-disjoint blue paths, disjoint of P_1, P_2 , between F_1 and F_2 . It follows that Condition 1 holds, as required. \square

It remains to show that, if Conditions 1 to 3 do not hold, there are at least four vertices which are either red or blue. Clearly, each vertex $v \in V(G) \setminus (F_1 \cup F_2)$ is either red or blue. Let l be the number of these vertices. Denote

$$\begin{aligned} A_1 &= V(F_1) \cap V(F_3) \\ A_2 &= V(F_1) \cap V(F_4) \\ A_3 &= V(F_2) \cap V(F_3) \\ A_4 &= V(F_2) \cap V(F_4). \end{aligned}$$

Consider the following claim.

Claim 46. *Suppose that $|A_i| \geq n/4 - 1 + k$ where $1 \leq k \leq 4$. Then either one of the Conditions 1 to 3 from above holds or A_i contains at least k vertices which are either blue or red.*

Suppose that the claim holds and the above conditions do not. If there are at most three red or blue vertices, then the total number of vertices is at most $4(n/4 - 1) + (3 - l) + l \leq n - 1$, a contradiction. The proof of Lemma 44 thus follows from the proof of the aforementioned claim.

Proof of Claim 46. Without loss of generality, $i = 1$. Note that if $G[A_1, A_4]$ has a matching of size 7, Condition 1 holds. Indeed, let M be such a matching. If at least two edges in M are red and at least two are blue, Condition 1 holds, since each edge connects F_1 and F_2 as well as F_3 and F_4 . Thus we may assume that at most one edge is red.

Recall that there are vertex-disjoint red paths P_1, P_2 between F_3 and F_4 (otherwise Condition 3 holds). We assume that their interiors do not intersect $V(F_3 \cup F_4) = V(F_1 \cup F_2)$, so each of P_1 and P_2 intersects at most two edges of M . It follows that there exist blue edges $e_1, e_2 \in M$ that are disjoint of P_1 and P_2 , so Condition 1 holds.

It remains to consider the case where $G[A_1, A_4]$ has no matching of size 7. We deduce by König's theorem that there is a set $U \subseteq A_1 \cup A_4$ of at most six vertices which intersects each of the edges of $G[A_1, A_4]$. Note that by the minimum degree conditions, every vertex in A_4 has at least k neighbours in A_1 . Hence, $|U \cap A_1| \geq k$. Denote $A'_4 = A_4 \setminus U$. Then

$$e(G[A_1, A'_4]) \geq k(|A_4| - 6).$$

We conclude that at least k vertices in $U \cap A_1$ have at least $n/25$ neighbours in A_4 . Indeed, otherwise we have

$$e(G[A_1, A'_4]) \leq (k - 1)|A_4| + 6n/25 < k(|A_4| - 6),$$

a contradiction. Each of the vertices in A_1 with at least $n/25$ neighbours in A_4 is either red or blue. □

□

13 Almost spanning blue

In this section, we resolve Condition 1 from Lemma 31, which is the last remaining case.

Lemma 47. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F is a blue (α, k) -strongly-robust subgraph on at least $(1 - \varepsilon)n$ vertices. Then $V(G)$ may be partitioned into a blue cycle and a red cycle.*

To prove Lemma 47, we extend F to include all vertices that send a fairly large number of blue edges into F , and denote the set of remaining vertices by Z . We consider two cases, according to the size of Z .

If Z is large, the reduced graph can easily be seen to have a perfect matching consisting of a connected blue matching and a connected red matching. We have to be slightly more careful than usual when obtaining cycles from the connected matching so as to cover Z . The leftover vertices of F can be absorbed as usual.

If Z is small, we apply the regularity lemma to F and prove a structural result on the reduced graph, focusing on ways to obtain perfect matchings. In each of the cases for the structure of the reduced graph, we can partition almost all of the vertices into a red cycle and a blue one. The vertices of Z may be covered using their large degree and the leftover vertices of F may be absorbed as usual.

Proof of Lemma 47. Let F_1 be the graph obtained by adding to F the vertices in G with at least αn blue neighbours in F , so F_1 is $(\alpha^3/2, k+2)$ -strongly-robust by Lemma 5. Let $Z = V(G) \setminus V(F_1)$, then

$$\deg_R(z) \geq (3/4 - \varepsilon - \alpha)n \geq (3/4 - 2\varepsilon)n \text{ for every } z \in Z.$$

It follows that every two vertices in Z have at least $(1/2 - 4\varepsilon)n$ red neighbours in common. We consider two cases according to the size of Z . Pick $\beta > 0$ by Lemma 3 so that F_1 remains $(\alpha^3/4, k+2)$ -strongly-robust after removing at most βn vertices.

Case 1: $|Z| \geq \beta n$

Let Q be a $\rho^2 n$ -absorbing path in F_1 of length at most ρn . Pick a suitably small η , and let Γ be the reduced graph obtained by applying the regularity lemma to G with parameters η and $d = 6\eta$. As usual, $\delta(\Gamma) \geq (3/4 - 2\varepsilon)m$, where $m = |\Gamma|$.

Let Φ_1 be the blue subgraph spanned by the clusters in $V(F_1)$. Then $|\Phi_1| \geq (1 - 2\varepsilon)m$ and Φ_1 is connected by Lemma 17. The vertices in clusters contained in Z have red degree at least $(3/4 - 4\varepsilon)m$ in Γ (note that since $|Z| \geq \beta n \geq 2\eta n$, there are such clusters). In particular, Γ has a red component Φ_2 of order at least $(3/4 - 4\varepsilon)m$. It is easy to check that $\Phi_1 \cup \Phi_2$ has a perfect matching M , e.g. by Theorem 19.

Let U_1 be the set of vertices in clusters of the blue edges in M and let U_2 be the set of vertices in clusters of the red edges. Note that U_1 does not intersect Z (since Φ_1 does not contain clusters in Z). We obtain the required partition into a blue cycle and a red cycle as follows.

Suppose first that $|U_2| \leq 3n/8$. Fix two vertices $z_1, z_2 \in Z$. Let W be a set consisting of 3η of the vertices of each cluster in $V(F_1)$. Clearly, z_1 and z_2 each have many neighbours in the set of vertices belonging to the clusters of Φ_2 . Thus, by Lemma 18, there is a red path P_1 in G' between z_1 and z_2 spanning at least $(1 - 6\eta)$ of the vertices of $U_2 \setminus W$ and using at most m^2 other vertices.

Denote by Z' the set of vertices in Z which are not covered by this path. Note that $|Z'| \leq 9\eta n$. Furthermore, every two vertices of $Z' \cup \{z_1, z_2\}$ have at least $n/16$ common red neighbours in $V(G') \setminus U_2$. It is thus possible to find a path in $G' \setminus (V(P_1) \cup U_1)$ between z_1 and z_2 , containing Z and using at most 200η of the vertices of each cluster. Indeed, such a path can be constructed greedily. Suppose $Z' = \{z_3, \dots, z_t\}$. For $i = 2, \dots, t$ we find a common red neighbour of z_i, z_{i+1} (where subscripts are taken modulo t) which was not used before and which does not belong to a cluster in U_2 with at least 200η of its vertices already used. In each step, there are at least $n/16$ possible choices, out of which at most $9\eta n$ were already used, and at most $9\eta n/200\eta < n/20$ belong to clusters for which at least 200η of its vertices are used. Thus it is possible to choose a suitable vertex.

We now construct a blue cycle which is disjoint of C , contains Q and misses at most 210η of the vertices of each cluster in U_1 , using Lemma 18 (we use the vertices of W to connect cluster pairs). The missing vertices can be absorbed by Q , completing the desired cycle partition.

Suppose now that $|U_2| \geq 3n/8$. By Lemma 18, there is a blue cycle C_1 and a red cycle C_2 , which are disjoint, and C_i covers all but at most 9η of the vertices of U_i . In particular, the red cycle has length at least $5n/16$. Let Z' be the set of vertices of Z which are not covered by either of the cycles. We show how to obtain a red cycle C'_2 such that $Z \subseteq V(C'_2) \subseteq V(C_2) \cup Z$ and $|V(C'_2) \setminus V(C_2)| \leq 60\eta n$. To that end, we claim that the vertices of Z' can be inserted one by one, such that in each step at most 20 of the original vertices of C are removed, none of them from Z . If z cannot be inserted as explained, the number of red neighbours of z in the cycle is at most $40|Z'| + n/20$. But every vertex $z \in Z'$ has at least $(1/16 - 3\varepsilon)n$ red neighbours in the cycle obtained from C_2 , as long as it has length at least $(5/16 - 60\eta)n$, implying that z may be inserted. There are at most $20|Z'| + 9\eta n \leq \rho^2 n$ vertices missing from $V(C_1) \cup V(C'_2)$, all of them from $V(F_1)$. They can be absorbed by Q .

Case 2: $|Z| \leq \beta n$

Let P be a path that alternates between Z and F_1 and covers all but three vertices in Z (if $|Z| < 3$, we take P to be empty). Denote by Z' the set of vertices in Z that are uncovered by P . Let $F_2 = F_1 \setminus V(P)$, so F_2 is $(\alpha/4, k+2)$ -strongly-robust by the choice of β . Apply Lemma 7 to obtain a $\rho^2 n$ -absorbing path Q in F_2 of length at most ρn , and denote $F_3 = F_2 \setminus V(Q)$. F_3 is $(\alpha^3/8, k+2)$ -strongly-robust, by Lemma 3. Furthermore, $|F_3| \geq (1 - 2\varepsilon)n$ and the vertices of Z' have at least $(3/4 - 4\varepsilon)n$ red neighbours in F_3 .

Apply the regularity lemma to the graph $G[V(F_3)]$ with a suitably small parameter η . Let Γ be the corresponding $(\eta, 4\eta)$ -reduced graph. We have $\delta(\Gamma) \geq (3/4 - 2\varepsilon)m$ and by Lemma 17, Γ_B is connected. We shall use the following proposition.

Proposition 48. *One of the following assertions holds.*

1. Γ_B has a perfect matching;

2. *There is a red component Φ of order at least $(1/2 - 30\varepsilon)m$ such that $\Gamma_B \cup \Phi$ has a perfect matching;*
3. *There are disjoint subsets $X_1, X_2 \subseteq V(G)$ of size at least $(1/4 - 80\varepsilon)m$ such that*
 - (a) *$X_1 \cup X_2$ is independent in Γ_B ;*
 - (b) *$\Gamma_R[X_i]$ is connected and $\Gamma_B \cup \Gamma_R[X_i]$ has a perfect matching for $i \in [2]$;*
4. *There exist disjoint subsets X_1, X_2, Y_1, Y_2 of size at least $(1/4 - 200\varepsilon)m$ such that*
 - (a) *$\Gamma_R[X_i \cup Y_i]$ is connected and non-bipartite for $i \in [2]$;*
 - (b) *$\Gamma_B[X_1 \cup X_2]$ and $\Gamma_B[Y_1 \cup Y_2]$ are connected.*

Before proving Proposition 48, we show how to complete the proof of Lemma 47 using the proposition.

Perfect matching in Γ_B

Suppose that Γ_B has a perfect matching. Recall that Γ_B is connected, so this matching is connected. We complete $Z' \cup V(P)$ to a red cycle C_1 using at most four additional vertices of F_3 (recall that P is the red path defined at the beginning of the subsection). By Lemma 18, there exists a blue cycle C_2 , disjoint of C_1 which extends the absorbing path Q and contains all but at most $6\eta n \leq \rho^2 n$ vertices of F_3 . The remaining vertices can be absorbed by Q to obtain a blue cycle C'_2 . The cycles C_1, C'_2 form the required cycle partition.

Perfect matching in union of Γ_B with a half-sized red component

Suppose that Φ is a red component of size at least $(1/2 - 30\varepsilon)m$ and $\Gamma_B \cup \Phi$ has a perfect matching. First extend P to a path P' that contains Z' and has both ends in Z' (we can do this using at most six additional vertices; if Z' is empty we take P' to be empty and if Z' consists of a single vertex we let P' be that vertex). Note that every vertex of Z' sends many red edges to the set of vertices contained in the clusters defined by $V(\Phi)$. By Lemma 18, there exist vertex-disjoint cycles C_1, C_2 such that C_1 is blue and contains the absorbing path Q and C_2 is red and contains the path P' . Furthermore, the cycles C_1, C_2 cover all but at most $\rho^2 n$ vertices belonging to F_3 , which may be absorbed by Q , completing the desired partition into a red cycle and a blue one.

Perfect matching in union of Γ_B with either of two quarter-sized red components

Let $X_1, X_2 \subseteq V(\Gamma)$ be disjoint sets of size at least $(1/4 - 80\varepsilon)m$ satisfying the following conditions.

- $X_1 \cup X_2$ is independent in Γ_B ;

- $\Gamma_R[X_i]$ is connected and $\Gamma_B \cup \Gamma_R[X_i]$ has a connected matching for $i \in [2]$.

Note that we may assume that $\Gamma_R[X_1 \cup X_2]$ is not connected, since otherwise we may proceed as in the previous case.

Let U_i be the set of vertices contained in the clusters in X_i . Note that $|U_i| \geq (1/4 - 81\varepsilon)n$. We define a path P' as follows. If $|Z'| = 3$, denote $Z' = \{z_1, z_2, z_3\}$. Then without loss of generality, z_1, z_2 send at least $2\eta n$ red edges into U_1 . Extend P to a path P' containing Z' with ends z_1, z_2 (using at most three additional vertices). If $|Z'| = 1$, denote $Z = \{z\}$ and take $P' = (z)$, and suppose without loss of generality that z has at least $2\eta n$ red neighbours in U_1 . If $Z = \emptyset$, take P' to be the empty path. As before, since $\Gamma_B \cup \Gamma_R[X_1]$ has a perfect matching, Lemma 18 implies that there exist disjoint cycles C_1, C_2 such that C_1 is blue and contains Q and C_2 is red and contains P' , and together they cover all but at most $\rho^2 n$ vertices of F_3 , which may be absorbed by C_1 .

It remains to consider the case where $|Z'| = 2$. Denote $Z' = \{z_1, z_2\}$. If for some $i \in [2]$, both z_1, z_2 have at least $2\eta n$ red neighbours in U_i , we may continue as before by taking P' to be a path of length 2 connecting z_1, z_2 . Thus we assume that

$$\deg_R(z_1, U_2), \deg_R(z_2, U_1) \leq 2\eta n \leq \varepsilon n.$$

Since $\deg_R(z_i) \geq (3/4 - 4\varepsilon)n$, it follows that $|U_1|, |U_2| \leq (1/4 + 5\varepsilon)n$. Let W be the set of common red neighbours of z_1 and z_2 in $V(G) \setminus (U_1 \cup U_2)$. Then $|W| \geq (1/2 - 20\varepsilon)n$.

Suppose that there is a vertex $w_1 \in W$ with at least $4\eta n$ red edges into $U_1 \cup U_2$. Without loss of generality, w_1 has at least $2\eta n$ red neighbours in U_1 . Take $P' = (w_1 z_2 w_2 z_1)$ for some $w_2 \in W$ and continue as before (when $|Z'| \neq 2$) to conclude that the desired partition exists.

We now assume that every vertex in W has at most εn red neighbours in $U_1 \cup U_2$. Furthermore, by the definition of the reduced graph, every vertex in $U_1 \cup U_2$ has at most $9\eta n \leq \varepsilon n$ red neighbours in W . Note that since $e(\Gamma[X_1, X_2]) = 0$, the graph $G_B[U_1 \cup U_2, W]$ is almost complete. It follows from Lemma 29 (with parameter 162ε) that the required cycle partition exists.

Four half-sized monochromatic components

Suppose that X_1, X_2, Y_1, Y_2 are disjoint sets of size at least $(1/4 - 200\varepsilon)m$ with the following properties.

- $\Gamma_R[X_i \cup Y_i]$ is connected and non-bipartite for $i \in [2]$;
- $\Gamma_B[X_1 \cup X_2]$ and $\Gamma_B[Y_1 \cup Y_2]$ are connected.

We conclude from Lemma 16 that there exist sets $U_1, U_2, W_1, W_2 \subseteq V(G)$ of order at least $(1/2 - \theta)n$, where $\theta = 202\varepsilon$, such that

- U_1, U_2 are disjoint, W_1, W_2 are disjoint and $U_1 \cup U_2 = W_1 \cup W_2$;
- $|U_i \cap W_j| \geq (1/4 - \theta)n$ for $i, j \in [2]$;
- $G_B[U_i]$ is (γ, l) -robust and $G_R[W_i]$ is (γ, l) -strongly-robust for $i \in [2]$, where $\gamma = \gamma(\varepsilon, \alpha, k)$ and $l = l(\varepsilon, \alpha, k)$.

By Lemma 44, $V(G)$ may be partitioned into a red cycle and a blue one. This completes the proof of Lemma 47, under the assumption that Proposition 48 holds. We prove it in the following subsection. \square

Proof of Proposition 48

In this subsection, we prove Proposition 48. We shall consider four cases according to the sizes of the red components in Γ . In each of these cases, we apply Lemmas 25 or 26 to gain additional structural information about the graph Γ if Γ_B has no perfect matching (otherwise we are done).

Proof of Proposition 48. The following claim reduces the proof to the following three cases.

Claim 49. *One of the following conditions holds.*

1. Γ_R has a component of order at least $(1/2 - 8\varepsilon)m$;
2. There is a tripartition $\{X_1, X_2, X_3\}$ of $V(\Gamma)$ such that $|X_i| \leq (1/2 - 8\varepsilon)m$ and every red component is contained in one of the sets X_i ;
3. There is a partition $\{X_1, X_2, X_3, X_4\}$ of $V(\Gamma)$ such that $|X_i| \geq (1/4 - 20\varepsilon)m$ and every red component is contained in one of the X_i 's.

Proof. Let $\{X_1, X_2, X_3\}$ be a partition of $V(\Gamma)$, such that every red component is in one of the parts, $|X_1| \geq |X_2| \geq |X_3|$ and $|X_1|$ is minimised under these conditions (and $|X_2|$ is minimised under these conditions).

We may assume that $|X_1| > (1/2 - 8\varepsilon)m$ (otherwise the second condition holds). It follows that $|X_3| \leq (1/4 + 4\varepsilon)m$. We conclude that X_1 has no red component of order at most $(1/4 - 12\varepsilon)m$. Indeed, suppose that U is the vertex set of such a component. Then, by considering $\{X_1 \setminus U, X_2, X_3 \cup U\}$, we reach a contradiction to the choice of X_1, X_2, X_3 .

We may also assume that X_1 consists of at least two components (otherwise the first condition holds). It follows that $|X_2| + |X_3| > (1/2 - 8\varepsilon)m$; indeed, otherwise we obtain a contradiction to the choice of X_i by considering the partition $\{X_1 \setminus U, U, X_2 \cup X_3\}$, where U is the vertex set of some component in X_1 .

So X_1 consists of two red components. Let U_1 and U_2 be their vertex sets, and suppose that $|U_1| \geq |U_2|$. So $|U_1|, |U_2| \geq (1/4 - 12\varepsilon)m$ and $|U_2| \leq (1/4 + 4\varepsilon)m$. Hence $|X_2|, |X_3| \geq (1/4 - 12\varepsilon)m$;

indeed, otherwise consider $\{U_1, X_2, X_3 \cup U_2\}$. The partition $\{U_1, U_2, X_2, X_3\}$ satisfies the third condition. \square

We prove Proposition 48 in each of the three cases in Claim 49. The proof for the first case, where there is a large red component, is the longest and we leave it to the end of the proof.

Tripartition

In Case 2 of Claim 49, there exists a tripartition $\{X_1, X_2, X_3\}$ of $V(\Gamma)$ such that $|X_i| \leq (1/2 - 8\varepsilon)m$ and every red component is contained in one of the red sets X_i . We assume that Γ_B has no perfect matching. By Lemma 25, without loss of generality, there exist subsets $Y_1 \subseteq X_1, Y_2 \subseteq X_2$ such that $|Y_1|, |Y_2| \geq (1/4 - 10\varepsilon)m$ and $Y_1 \cup Y_2$ is independent in Γ_B .

It follows that

$$\delta(\Gamma_R[Y_i]) \geq |Y_i| - 12\varepsilon n. \quad (1)$$

In particular, $\Gamma_R[Y_i]$ is connected. We show that $\Gamma_B \cup \Gamma_R[Y_i]$ has a perfect matching for $i \in [2]$. Suppose to the contrary that $\Phi = \Gamma_B \cup \Gamma_R[Y_1]$ has no perfect matching. By Lemma 25, it follows that there exist subsets $Z_i \subseteq X_i$ and $Z_j \subseteq X_j$ of size at least $(1/4 - 10\varepsilon)n$ such that $Z_1 \cup Z_2$ is independent in Φ , for some $1 \leq i \neq j \leq 3$.

Suppose first that $i = 2$. Then the intersection of Y_2 and Z_2 is non-empty. Let $u \in Y_2 \cup Z_2$. Note that u has at most $(1/4 + 2\varepsilon)m$ non-neighbours (in Γ) in $X_1 \cup X_3$, at least $(1/4 - 10\varepsilon)m$ of which belong to Y_1 , so $|Z_1 \cap Y_1| \geq (1/4 - 22\varepsilon)m$, a contradiction to (1) and the assumption that $Z_1 \cup Z_2$ is independent in Φ .

It remains to consider the case $i = 1, j = 3$. As before, we may pick $u \in Y_1 \cap Z_1$. But u has at least $|Z_3| + |Y_2| \geq (1/2 - 20\varepsilon)m$ non-neighbours in Γ , a contradiction.

Almost equipartition into four parts

In Case 3 of Claim 49, $\{X_1, X_2, X_3, X_4\}$ is a partition of $V(G)$ such that every red component is contained in one of the parts X_i and $|X_i| \geq (1/4 - 12\varepsilon)m$.

Apply Lemma 26 to the graph Γ_B , with the bipartition $\{X_1 \cup X_2, X_3 \cup X_4\}$ and parameter 24ε . Assuming that Γ_B has no perfect matching, it is easy to verify that the first three conditions cannot hold. It follows that Γ_B has an independent set Y of size at least $(1/2 - 72\varepsilon)m$. This implies that $\delta(\Gamma_R[Y]) \geq (1/4 - 74\varepsilon)m$, implying that $\Gamma_R[Y]$ is the union of two red connected subgraphs of order at least $(1/4 - 74\varepsilon)m$. Without loss of generality, $Y \subseteq X_1 \cup X_2$. Denote $Y_i = Y \cap X_i$.

If $|X_1| + |X_2| \geq |X_3| + |X_4|$, it is not hard to deduce from Lemma 26 that $\Gamma_B \cup \Gamma_R[Y_i]$ has a perfect matching for $i \in [2]$. Otherwise, $\Gamma_B[X_1 \cup X_2, X_3 \cup X_4]$ is almost complete, and, again, by Lemma 26, if Γ_B has no perfect matching, $X_3 \cup X_4$ contains an independent set Z' of size at least $(1/2 - 72\varepsilon)m$. Denoting $Z_i = Z' \cap X_i$ for $i \in [3, 4]$, we conclude as before that $|Z_i| \geq (1/4 - 74\varepsilon)m$, $\Gamma_R[Z_i]$ is

connected and $\Gamma_B \cup \Gamma_R[Z_i]$ has a perfect matching for $i \in [3, 4]$. The third assertion of Proposition 48 holds.

Large red component

We now consider the last remaining case where Condition 1 of Claim 49 holds. Let Φ_1 be the largest red component, so it has order at least $(1/2 - 8\varepsilon)m$. Denote $X_1 = V(\Phi_1)$, $X_2 = V(\Gamma) \setminus X_1$ and $\Gamma_1 = \Gamma_B \cup \Phi_1$. By Lemma 26, either Γ_1 has a perfect matching, in which case we are done, or one of the following holds (note that if $|\Phi_1| \geq (1/2 + 4\varepsilon)m$ then Γ_1 has a perfect matching by Theorem 19, so we may assume that $|X_2| \geq (1/2 - 4\varepsilon)m$ and Lemma 26 is applicable with parameter 8ε).

1. There is a subset $A_2 \subseteq X_2$ of size at least $(1/4 - 32\varepsilon)m$ that is independent in Γ_1 and its neighbourhood in Γ_1 has size at most $(1/4 + 24\varepsilon)m$.
2. $|X_2| \geq |X_1|$ and X_2 contains an independent set (with respect to Γ_1) A of size at least $(1/2 - 8\varepsilon)m$.

If the first condition holds, then, in particular, X_2 contains a red component of order at least $(1/2 - 26\varepsilon)m$ (as any vertex in A_2 has red degree at least $(1/2 - 26\varepsilon)m$). If the second condition holds, then $\delta(G_R[A]) \geq (1/4 - 10\varepsilon)m$. It follows that either A consists of two red connected subgraphs of order at least $(1/4 - 10\varepsilon)m$ or $G_R[A]$ is connected. In the former case, we have that $G_B[X_1, X_2]$ is almost complete and $G_R[Y_i]$ is almost complete for $i \in [2]$, where Y_1 and Y_2 are the vertex sets of the two red components in A . It is easy to show that $G_B \cup G_R[Y_i]$ has a perfect matching for $i \in [2]$, hence the third condition of Proposition 48 holds.

It remains to consider the case where X_2 contains a red component Φ_2 of order at least $(1/2 - 26\varepsilon)m$. Denote $\Gamma_2 = G_B \cup \Phi_2$. Note that $|\Phi_1| \geq |\Phi_2|$, by our assumption that Φ_1 is the largest red component. Now, by applying Lemma 26 to Γ_1 with the bipartition $\{V(\Phi_2), V(\Gamma) \setminus V(\Phi_2)\}$ and parameter 30ε , we conclude that the following holds.

There is an independent set $B_2 \subseteq V(\Phi_2)$ of size at least $(1/4 - 120\varepsilon)m$ such that $|N_{\Gamma_1}(B_2)| \leq (1/4 + 90\varepsilon)m$.

Similarly, if Γ_2 has no perfect matching, one of the two following conditions holds.

1. $V(\Phi_1)$ contains an independent set (in Γ_2) Y_1 of size at least $(1/2 - 30\varepsilon)m$.
2. There is an independent set $B_1 \subseteq V(\Phi_1)$ of size at least $(1/4 - 120\varepsilon)m$ such that $|N_{\Gamma_2}(B_1)| \leq (1/4 + 90\varepsilon)m$.

In the latter case, denote $Z_1 = B_1 \cup N_R(B_2)$. Then $|Z_1| \geq (1/2 - 92\varepsilon)m$ and $\Gamma_R[Z_1]$ is connected and non-bipartite. Now let W_1 be Y_1 if the first condition holds, and Z_1 if the second condition

does. Denote

$$\begin{aligned} B_1 &= W_1 \cap N_{\Gamma_1}(B_2) \\ C_1 &= W_1 \setminus N_{\Gamma_1}(B_2) \\ W_2 &= N_{\Gamma_R}(B_2) \\ C_2 &= W_2 \cap N_{\Gamma_2}(B_1). \end{aligned}$$

It is easy to verify that $|B_i|, |C_i| \geq (1/4 - 182\varepsilon)m$, and that $\Gamma_R[B_i, C_i]$ is connected and non-bipartite for $i \in [2]$. Furthermore, $\Gamma_B[B_1, B_2]$ and $\Gamma_B[C_1, C_2]$ are connected. Hence Condition 3 of Proposition 48 holds, completing the proof in this case. \square

The proof of Lemma 47 concludes our proof of Theorem 1. We remind the reader that Lemmas 29 and 30 were used several times in the proof, and we have yet to prove them. The next two sections, Sections 14 and 15 are devoted to the proofs of Lemmas 29 and 30 respectively.

14 Proof of Lemma 29

In this section, we prove Lemma 29. Before turning to the proof, we remind the reader of the statement.

Lemma 29. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that $S, T \subseteq V(G)$ satisfy the following conditions.*

- S, T are disjoint and $|S|, |T| \geq (1/2 - \varepsilon)n$;
- $\delta(G_B[S, T]) \geq (1/4 - \varepsilon)n$;
- For every $S' \subseteq S, T' \subseteq T$ with $|S'|, |T'| \geq (1/4 - 100\varepsilon)n$, we have $e(G_B[S', T']) \geq 25\varepsilon n^2$.

Then $V(G)$ may be partitioned into a red cycle and a blue one.

The main tool we use in the proof is Lemma 27, which is a stability version of a special case of Corollary 20. Our aim would be to find a short red cycle C and a short blue path P , whose removal from G leaves a balanced bipartite graph. We then apply Lemma 27 to deduce that P may be extended to a blue cycle with vertex set $V(G) \setminus V(C)$.

Proof of Lemma 29. We start by modifying the sets S, T as follows.

$$\begin{aligned} S_1 &= S \cup \{v \in V(G) \setminus (S \cup T) : \deg_B(v, T) \geq 24\varepsilon n\}. \\ T_1 &= T \cup \{v \in V(G) \setminus (S_1 \cup T) : \deg_B(v, S) \geq 24\varepsilon n\}. \\ X &= V(G) \setminus (S_1 \cup T_1). \end{aligned}$$

Remark. The vertices in X have red degree at least $(3/4 - 50\varepsilon)n$ in G .

We will find a red cycle and a blue path with one end in S_1 and one in T_1 , which are disjoint, cover X and their removal from G leaves a balanced bipartite graph with a large number of vertices. We will then use the following claim to obtain the required partition into a blue cycle and a red one.

Claim 50. *Let $S' \subseteq S_1, T' \subseteq T_1$ be such that $|S'| = |T'| \geq (1/2 - 12\varepsilon)n$. Then $G_B[S', T']$ is Hamiltonian. Furthermore, for every $s \in S'$ and $t \in T'$, the graph $G_B[S', T']$ contains a Hamiltonian path with ends s and t .*

Proof. Denote $G' = G_B[S', T']$ and $Y = (S' \cup T') \setminus (S \cup T)$. We claim that there exists a path P of length at most $12\varepsilon n$ whose vertex set contains Y . Indeed, we may construct P greedily, by adding a vertex of Y one at a time. Suppose that we want to add the vertex $y_1 \in Y$ to a path P in G' of length at most $12\varepsilon n$, one of whose ends is $y_2 \in Y$. We may pick $z_1, z_2 \in ((S \cup T) \cap V(G')) \setminus V(P)$ such that z_i is a neighbour of y_i in G' . We have $N_{G'}(z_i) \geq (1/4 - 11\varepsilon)n$. In particular, by the third assumption of the lemma, there exists a path of length at most 2 between $N_{G'}(z_1)$ and $N_{G'}(z_2)$. Thus we may add y_1 to P using at most five additional vertices. Using this process, we obtain the desired path P , containing the vertices of Y . Denote by s, t the ends of P and assume that $s \in S, t \in T$ (we may need to extend P slightly).

Let G'' be the graph obtained from G' by removing the interior vertices of P and denote $n'' = |G''|$ and $\eta = 25\varepsilon$. Then $\delta(G'') \geq (1/4 - \eta)n''$. Let $S' \subseteq S, T' \subseteq T$ be subsets of size at least $(1/4 - 3\eta)n'' \geq (1/4 - 75\varepsilon)(1 - 20\varepsilon)n \geq (1/4 - 80\varepsilon)n$. Then by the assumptions of Lemma 29, $e(G'[S', T']) \geq \eta n^2$. By Lemma 27, it follows that G' is Hamiltonian. The same argument may be used to show that G' contains a Hamiltonian path with ends s, t for every $s \in S', t \in T'$. \square

We consider several cases, depending on the size of X and the behaviour of the vertices in X .

Case 1: $X = \emptyset$

Without loss of generality, $|S_1| \geq |T_1|$. Denote $k = |S_1| - |T_1|$; note that $k \leq 2\varepsilon n$. Suppose first that k is even. We use the following claim.

Claim 51. *The graph $G[S_1]$ either contains a blue path of length k , or it contains a red cycle of length k .*

Proof. Suppose that $G_B[S_1]$ has no path of length k . It follows from Erdős and Gallai's theorem, Theorem 21, that $e(G_B[S_1]) \leq kn/2 \leq \varepsilon n^2$. Setting $\eta = \sqrt{2\varepsilon}$, there are at most ηn vertices in S_1 of blue degree at least ηn in S_1 . Denote by U the set of vertices in S_1 with blue degree at most ηn . Then $\delta(G_R[U]) \geq (1/4 - 2\eta)n$.

We show that $G_R[U]$ contains a cycle of length k . We may assume that $k \geq 4$ because this assertion is trivial for $k \in \{0, 2\}$. Pick $u \in U$ and denote $A = N_R(u) \cap U$ (so $|A| \geq (1/4 - 2\eta)n$). If $G_R[A]$ has a path of length $k - 2$, together with the vertex u it forms a red cycle in S of length k .

Thus, we assume that $G_R[A]$ contains no path of length $k - 2$. It follows from Theorem 21 that $e(G_R[A]) \leq |A| \cdot \varepsilon n$. We deduce that at most $|A|/2$ vertices in A have red degree at least $4\varepsilon n$ in A . In particular, we may pick a set B of $k/2$ vertices in A with red degree at most $4\varepsilon n$ in A . For every $v \in B$ we have $\deg_R(v, U \setminus A) \geq (1/4 - 4\varepsilon - \eta)n$. It follows that every two vertices in B have at least say $n/8$ common red neighbours in $U \setminus A$. In particular, if $B = \{b_1, \dots, b_{k/2}\}$, we may pick distinct $c_1, \dots, c_{k/2} \in U \setminus A$ such that $(b_1, c_1, \dots, b_{k/2}, c_{k/2})$ is a red cycle in S of length k . \square

By Claim 51, either S_1 contains a blue path P of length k or it contains a red cycle C of length k . In the first case, it is easy to verify that P may be extended to a Hamilton cycle of G_B by Claim 50. Indeed, consider the bipartite graph $G_B[S_1 \setminus U, T_1]$ where U is the set of interior vertices of P_1 . This graph is almost balanced (namely the first set has one more vertex than the other), so it contains a Hamilton path whose ends are the ends of P . In the second case, we may conclude directly from Claim 50 that the graph $G_B \setminus V(C)$ is Hamiltonian.

We now suppose that k is odd. If S_1 contains a blue path P of length k , we continue as before. Otherwise, if S_1 contains a blue edge uv , we may find a red cycle C in $S_1 \setminus \{u, v\}$ of length $k - 1$, by the argument of Claim 50. It follows that the graph $G_B \setminus V(C)$ is Hamiltonian. Finally, if S_1 has no blue edges, we have $\delta(G_R[S_1]) \geq |S_1| - n/4 > |S_1|/2$, since $|S_1| \geq (n + 1)/2$. It follows from Bondy's theorem (22) that $G_R[S_1]$ is pancyclic, in particular it contains a cycle C of length k . We proceed as before to conclude that $G_B \setminus V(C)$ is Hamiltonian.

Case 2: $|X| = 1$

Denote $X = \{x\}$. Again, we assume that $|S_1| \geq |T_1|$ and denote $k = |S_1| - |T_1|$. Suppose first that k is odd. We shall use the following claim, whose proof is similar to the proof of Claim 51.

Claim 52. *Either $G[S_1]$ has a blue path P of length k , or $G[S_1 \cup \{x\}]$ has a red cycle C of length $k + 1$ containing x .*

Proof. If $G[S_1]$ contains no blue path of length k , we consider the set $A = N_R(x, S_1)$. Recall that $|A| \geq (1/4 - 25\varepsilon)n$. As in Claim 51, we conclude that either $G[A]$ contains a red path of length $k - 1$, or $G[A, S_1 \setminus A]$ contains a red path of length $k - 1$ with both ends in A . \square

If $G[S_1]$ contains a path of length k , the graph $G_B \setminus \{x\}$ is Hamiltonian and we may take (x) to be the red cycle. Otherwise, let C be a red cycle of length $k + 1$ in $S_1 \cup \{x\}$ containing x and we again find that $G_B \setminus V(C)$ is Hamiltonian by Claim 50.

We now consider the case where k is even. Note that if $k = 0$, we can take (x) to be the red cycle and $G_B[S_1, T_1]$ is Hamiltonian. Thus, we assume that $k \geq 2$. Denote $A = N_R(x, S_1)$.

Claim 53. *One of the following conditions holds.*

1. $G[S_1]$ has a blue path of length k .

2. $G[S_1 \cup \{x\}]$ contains a red cycle C of length k going through x and a blue edge e which is disjoint of C .
3. $G[S_1 \cup \{x\}]$ has a red cycle C of length $k + 1$ going through x .
4. $G[S_1]$ has no blue edges and $G[A]$ has no red edges.

Proof. It is easy to conclude, as in Claims 51 and 52, that if $G[S_1]$ has at least one blue edge, one of the first two conditions holds. Thus we assume that $G[S_1]$ has no blue edges. Assuming the fourth condition does not hold, we may further assume that $G[A]$ has a red edge uv . We prove that the third condition holds under these assumptions. As before, we may assume that $e(G_R[A]) \leq \varepsilon n^2$, because otherwise $G[A]$ contains a red path of length $k - 1$ and the third condition holds.

Note that we have $\delta(G_R[S_1]) \geq |S_1| - n/4 \geq n/4$. Thus we may construct a red path in $G[S_1]$ on $k - 2$ vertices $P = (v = v_1, \dots, v_{k-2})$. If there exists a vertex $v_{k-1} \in S_1$ which is a common red neighbour of v_{k-2} and x , we obtain the red cycle $C = (v_1, \dots, v_{k-1}, x, u)$. Otherwise, the sets $A' = A \setminus (V(P) \cup \{u\})$ and $B = N_R(v_{k-2}, S) \setminus (V(P) \cup \{u\})$ are disjoint. Note that $|A'|, |B| \geq (1/4 - 60\varepsilon)n$. Since $e(G_R[A']) \leq \varepsilon n^2$ and $\delta(G_R[S_1]) \geq n/4$, we conclude that $G_R[A', B]$ contains an edge $v_{k-1}v_k$ (where $v_{k-1} \in B, v_k \in A'$). It follows that (v_1, \dots, v_k, x) is a red cycle in S_1 of length k . \square

In each of the first three conditions of Claim 53, we may proceed as before to conclude that G has the desired partition into a red cycle and a blue one. Thus we may assume that S_1 spans no blue edges and that A spans no red edges. In particular, $\deg_R(x, S_1) \leq n/4$, hence x has at least two blue neighbours in S_1 , since $\deg(x, S_1) \geq |S_1| - (n/4 - 1) \geq n/4 + 1/2 + k/2 \geq n/4 + 3/2$. We consider the cases $k = 2$, $k = 4$ and $k \geq 6$ separately.

Suppose first that $k = 2$. Let $u, v \in S_1$ be two blue neighbours of x in S_1 . Let C_1 be a (red) cycle consisting of a single vertex in $S_1 \setminus \{u, v\}$. We may find a blue cycle spanning $V(G) \setminus V(C_1)$ by Claim 50.

Suppose now that $k = 4$. We have $|S_1| = n/2 + 3/2$. Thus x has at least three blue neighbours $u, v, w \in S_1$. Pick an edge ab in S_1 such that a and b are distinct from u, v, w . Since $\deg_R(a, S_1) \geq |S_1| - n/4 \geq n/4 + 3/2$, we conclude that a and b have a common red neighbour $c \in S_1$. Let C_1 be the red triangles (abc) . Without loss of generality $c \neq u, v$. We proceed as before, to show that the graph $G_B \setminus \{a, b, c\}$ is Hamiltonian.

It remains to consider the case $k \geq 6$. Fix u, v to be blue neighbours of x in S_1 and denote $S_2 = S_1 \setminus \{u, v\}$. Note $|S_2| \geq n/2 + 1/2$ and $\delta(G_R[S_2]) \geq |S_2| - n/4 > |S_2|/2$. It follows from Theorem 22, that $G_R[S_2]$ is pancyclic. In particular, it contains a red cycle of length $k - 1$. We proceed as before.

From now on, we may assume that $|X| \geq 2$. Set $\eta = 3\sqrt{\varepsilon}$.

Case 3: there exist $x_1, x_2 \in X$ with $\deg_R(x_1, S), \deg_R(x_2, T) \geq (1/4 + 10\eta)n$

Denote $X = \{x_1, \dots, x_r\}$. Recall that $\deg_R(x_i) \geq (3/4 - 50\varepsilon)n$. Thus we may pick distinct $y_i \in V(G) \setminus X$ such that y_i is a common red neighbour of x_i and x_{i+1} for $i \in [2, r-1]$. Denote $Y = \{y_2, \dots, y_{r-1}\}$. Let $S_2 = S_1 \setminus Y$ and $T_2 = T_1 \setminus Y$.

Claim 54. *The graph $G[S_2]$ either contains a blue path of length $5\varepsilon n$ or for every $2 \leq l \leq 5\varepsilon n$ it contains a red path of length $l-1$ whose one end is a red neighbour of x_1 and the other is a red neighbour of x_r .*

Similarly, the graph $G[T_1]$ either contains a blue path of length $5\varepsilon n$ or for every $2 \leq l \leq 5\varepsilon n$ it contains a path of length $l-1$ a red neighbour of x_1 as one end and a red neighbour of x_2 as the other end.

Proof. We prove the first part of Claim 54; the second part of follows similarly. Suppose that $G[S_2]$ has no blue path of length $5\varepsilon n$. It follows by Theorem 21 that $e(G[S_2]) \leq 3\varepsilon n^2$. Denote by U the set of vertices in S' with blue degree at most ηn (recall that $\eta = 3\sqrt{\varepsilon}$). Then $|S_2 \setminus U| \leq \eta n$ and $\delta(G_R[U]) \geq (1/4 - 3\eta)n$. Pick $u \in N_R(x_r, U)$. Greedily construct a red path $P = (u = u_1, \dots, u_{l-1})$ in U . Denote $A = N_R(x_1, U \setminus V(P))$ and $B = N_R(u_{l-1}, U \setminus V(P))$. As $|Y| + |P| \leq \eta n$, we have $|A| \geq (1/4 + 6\eta)n$ (by the assumption on x_1) and $|B| \geq (1/4 - 4\eta)n$. Thus $|A \cap B| \geq |A| + |B| - |U| \geq \eta n$. It follows that we may pick $u_l \in A \cap B$. The path $(x_r, u_1, \dots, u_l, x_1)$ satisfies the requirements of Claim 54. \square

Without loss of generality, we assume that $|S_2| \geq |T_2|$. Denote $k = |S_2| - |T_2|$, so $k \leq 4\varepsilon n$. It is easy to conclude from Claim 54 that we may find vertex-disjoint paths $P_1 \subseteq G_B[S_2]$ and $Q_1 \subseteq G_R[S_2]$ such that the following holds.

- Either P_1 has length k and Q_1 is a singleton, or P_1 is the empty path and Q_1 has length k .
- One end of Q_1 is a red neighbour of x_1 and the other end is a red neighbour of x_r (if Q_1 is a singleton, then it is a common red neighbour of x_1, x_r).

Indeed, if $G[S_2]$ contains a blue path of length $5\varepsilon n$, we may find such P_1, Q_1 where P_1 has length $k-1$ and Q_1 is any common red neighbour of x_1, x_r in S_2 (note that such a common neighbour exists). Otherwise, $G[S_2]$ contains a red path of length $k-1$ whose ends are a red neighbour of x_1 and a red neighbour of x_2 .

Similarly, we may pick P_2, Q_2 to be a blue and a red path in $G[T_2]$ as follows.

- Either P_2 is the empty path and Q_2 has length 1 or P_2 has length 1 and Q_2 is a singleton.
- Q_2 has one end which is a red neighbour of x_1 and the other is a red neighbour of x_2 (if Q_2 is a singleton then it is a common red neighbour of x_1 and x_2).

We take C to be the red cycle $(Q_1x_1Q_2x_2y_2x_3\dots x_l)$. The paths P_1, P_2 may be extended to a Hamilton cycle of $G_B \setminus V(C)$ by Claim 50.

Without loss of generality, we may now assume the following.

$$\deg_R(x, T) \leq (1/4 + 10\eta)n \text{ for every } x \in X. \quad (2)$$

It follows that $\deg_R(x, S) \geq (1/2 - 11\eta)n$ for every $x \in X$.

Case 4: some $x_1, x_2 \in X$ have at least $3\eta n$ common red neighbours in T

We proceed similarly to the previous case. For $2 \leq i \leq r-1$, pick y_i to be a common red neighbour of x_i and x_{i+1} such that the y_i 's are distinct and do not belong to X .

Let $S_2 = S_1 \setminus Y$ and $T_2 = T_1 \setminus Y$. Note that $\deg_R(x_i, S_1) \geq (1/2 - 11\eta)n$ for $i \in [r]$ by (2). Consider the set D of common red neighbours of x_1, x_r in S' . Then $|D| \geq (1/2 - 23\eta)n$. Clearly, D has either a blue path or a red one of length $5\epsilon n$.

Claim 55. *The graph $G[T_2]$ either has a blue path of length $5\epsilon n$ or it contains a red path of length l with ends which are neighbours of x_1, x_2 respectively, for every even $2 \leq l \leq 5\epsilon n$.*

Proof. Suppose that $G[T_2]$ has no blue path of length $5\epsilon n$. Denote by U the set of vertices of T_2 with at most ηn blue neighbours. Then $|T_2 \setminus U| \leq \eta n$ and $\delta(G_R[U]) \geq (1/4 - 3\eta)n$. Denote $A = N_R(x_1, U)$ and $B = N_R(x_2, U)$. Note that $|A|, |B| \geq (1/4 - 2\eta)n$ and $U \cap A \cap B \neq \emptyset$, by the assumptions on x_1 and x_2 ; let $u_1 \in U \cap A \cap B$.

If $|A \cup B| \geq (1/4 + 10\eta)n$, we may find a red path of length l as follows. Greedily pick a red path $P = (u_1, \dots, u_l)$ in U . As in Claim 54, there exists $u_{l+1} \in (A \cup B) \setminus V(P)$ which is a red neighbour of u_l . The path (u_1, \dots, u_{l+1}) satisfies the requirements.

If $|A \cup B| \leq (1/4 + 10\eta)n$, it follows that $|A \cap B| \geq |A| + |B| - |A \cup B| \geq (1/4 - 14\eta)n$. If $A \cap B$ contains a red path of length l , we are done. Otherwise, we may continue as in Claim 51 to conclude that the graph $G_R[A \cap B, U \setminus (A \cap B)]$ has a path of length l with ends in $A \cap B$. \square

Denote $k = |S_2| - |T_2|$. Pick $2 \leq l_1, l_2 \leq 5\epsilon n$ such that l_2 is even and $l_1 - l_2 = k$. Similarly to the previous case, we may pick vertex-disjoint paths $P_1 \in G_B[S_2]$, $Q_1 \in G_R[S_2]$, $P_2 \in G_B[T_2]$ and $Q_2 \in G_R[T_2]$ with the following properties.

- One end of Q_1 is a red neighbour of x_1 and the other end is a red neighbour of x_r . Similarly, one of the ends of Q_2 is a red neighbour of x_2 and the other is a red neighbour of x_1 .
- Either $P_i = \emptyset$ and Q_i has length l_i or P_i has length l_i and Q_i is a singleton, for $i \in [2]$.

As before, we take C to be the red cycle $(Q_1x_1Q_2x_2y_2x_3 \dots x_r)$. The paths P_1, P_2 may be extended to a Hamilton cycle in $G_B \setminus V(C)$.

Note that if $|X| \geq 3$, one of Cases 3 and 4 holds (perhaps with the roles of S and T reversed). Thus we may assume that $|X| = 2$, and denote $X = \{x_1, x_2\}$.

Case 5: $\deg_R(x_i, T) \leq (1/4 + 10\eta)n$ and $|N_R(x_1, T) \cap N_R(x_2, T)| \leq 3\eta n$

Denote

$$A = N_R(x_1, T_1), B = N_R(x_2, T_1) \text{ and } D = N_R(x_1, S_1) \cap N_R(x_2, S_1).$$

Note that $|D| \geq (1/2 - 23\eta)n$.

If $|S_1| \geq |T_1| + 2$, denote $k = |S_1| - |T_1|$, so $2 \leq k \leq 2\epsilon n$. As usual, $G[D]$ either contains a blue path P or a red path Q of length $k - 2$. In the former case, pick C to be a 4-cycle consisting of x, y and two vertices in $D \setminus V(P)$. In the latter case, extend Q to a cycle C through x, y using an additional vertex of D . For convenience denote $P = \emptyset$. In both cases, the path P may be extended to a Hamilton cycle of $G_B \setminus V(C)$. From now on, we may assume the following.

$$|S_1| \leq |T_1| + 1.$$

Denote $k = |T_1| - |S_1|$ (so $-1 \leq k \leq 2\epsilon n$). A *path forest* is a collection of vertex-disjoint paths. If $G_B[T_1]$ contains a path forest H with $k + 2$ edges, we can finish the proof as follows. Pick C to be any red 4-cycle consisting of x_1, x_2 and two vertices from D . Then $G_B \setminus V(C)$ has a Hamilton cycle extending H . This can be seen by connecting the paths of H with paths in $G_B[S_1, T_1]$ of length at most 6 and using Claim 50. Thus we may assume the following.

$$G_B[T_1] \text{ has no path forest with } k + 2 \text{ edges. In particular, } e(G_B[T_1]) \leq \epsilon n^2. \quad (3)$$

Suppose that $G_R[T_1]$ contains a path P with one end in A and the other in B , of length l , where $\max\{0, k\} \leq l \leq 3\epsilon n$ (by a path of length 0 we mean a single vertex). Then we may find the desired cycle partition as follows. $G[D]$ contains a path Q of length $l - k$ which is either red or blue. If Q is red, take C to be the red cycle (x_1Px_2Q) , and the remaining graph $G_B \setminus V(C)$ is Hamiltonian. If Q is blue, take $C = (x_1Px_2u)$ where $u \in D \setminus V(Q)$. The leftover graph $G_B \setminus V(C)$ has a Hamilton path extending Q . Thus from now on we assume the following.

$$G_R[T_1] \text{ has no path of length } l, \text{ where } \max\{0, k\} \leq l \leq 3\epsilon n, \text{ with ends in } A \text{ and } B. \quad (4)$$

In particular, $e(G_R[A, B \setminus A]) \leq \epsilon n^2$, implying that $G_R[A], G_R[B]$ are almost complete (recall that $A \cap B$ have a small intersection and that T_1 spans few blue edges). Suppose that x_1 has two blue neighbours $u, v \in T_1$. It follows from the previous assumption that $G_R[B]$ contains a path P on $k + 1$ vertices. Form a red cycle C by adding the vertex x_2 to P . Then the graph $G_B \setminus V(C)$ is

Hamiltonian, since the graph $G_B \setminus (V(C) \cup \{x_1\})$ has a Hamilton path with ends u, v . Thus we may assume that both x_1 and x_2 have at most one blue neighbour in T_1 . In particular,

$$|A|, |B| \geq |T_1| - n/4. \quad (5)$$

We can now finish the proof if $k \in \{-1, 0\}$. Note that we have $e(G_B[T_1]) \leq 1$ by Assumption (3). It is easy to conclude, using Assumption (5) that one of the following conditions holds.

- $A \cap B \neq \emptyset$.
- $A \cap B = \emptyset$ and $e(G_R[A, B]) \geq 1$.
- There exists a vertex $u \in T_1$ which has red neighbours in both A and B .

It follows that there exists a red path in T_1 with one end in A and the other in B of length at most 2, contradiction Assumption (4). Thus the desired monochromatic cycle partition exists.

We may now assume $k = |T_1| - |S_1| \geq 1$. Denote $Y = \{u \in T_1 : \deg_B(u, T_1) \geq \eta m\}$. Then

$$|Y| \leq (k + 1)/2 \quad (6)$$

because otherwise $G_B[T_1]$ contains at least $(k + 2)/2$ vertex-disjoint paths of length 2, contradicting Assumption (3).

Denote $A' = A \setminus Y$ and $B' = B \setminus Y$. If there exists a path of length at most 2 in $G_R[T_1]$ with one end in A' and the other in B' , this path may be extended to a red path between A' and B' of length $k + 2$, contradicting assumption (4). Thus we assume the following.

$$A' \cap B' = \emptyset. \quad (7)$$

$$e(G_R[A', B']) = 0. \quad (8)$$

$$\text{No vertex in } T_1 \text{ has red neighbours in both } A' \text{ and } B'. \quad (9)$$

It is not hard to reach a contradiction from here, thus finishing the proof. By (7), $A \cap B \subseteq Y$. In particular,

$$|Y| \geq |A \cap B| \geq |A| + |B| - |T_1| \geq |T_1| - n/2 = (k - 2)/2. \quad (10)$$

Also, by (6),

$$|A'|, |B'| \geq |T_1| - n/4 - |Y| \geq n/4 - 3/2. \quad (11)$$

If $|Y| \leq (k - 1)/2$, it follows that $|A'|, |B'| \geq n/4 - 1/2$. By the minimum degree condition, we have that $\delta(G[A', B']) \geq 1$, and it is easy to deduce that $G[A', B']$ has a (blue) path forest on at least four edges. By (10), we may complete this into a blue path forest in T_1 with at least $k + 2$ edges, a contradiction to (3).

Thus we assume that $|Y| \geq k/2$. It follows from (3) that

$$e(G_B \setminus Y) \leq 1. \quad (12)$$

If $|A'| > n/4 - 1$, the graph $G[A', B']$ has at least two (blue) edges, a contradiction. Thus we assume that $|A'|, |B'| \leq n/4 - 1$. It follows that $|A \cup B| \leq |A'| + |B'| + |Y| < |T_1|$. Pick $u \in T_1 \setminus (A \cup B)$. By (11), (8), (12), all but at most two vertices of $A' \cup B'$ are connected to u , contradicting (9). \square

15 Proof of Lemma 30

In this section, we prove Lemma 30. We first remind the reader of the statement.

Lemma 30. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that there exists a partition $\{S, T, X\}$ of $V(G)$ with the following properties.*

- $|S|, |T| \geq (1/2 - \varepsilon)n$;
- $|X| \leq 2$ and if $|X| = 2$, there exists $u \in X$ such that $\deg_R(x, S) \leq \varepsilon n$ or $\deg_R(x, T) \leq \varepsilon n$;
- The sets S and T belong to different components of $G_R \setminus X$.

Then $V(G)$ may be partitioned into a red cycle and a blue one.

The idea of the proof is as follows. By Lemma 29, we may assume that there exist subsets $S' \subseteq S, T' \subseteq T$ of size almost $n/4$ such that $G[S', T']$ is close to being empty, implying that the subgraphs $G_B[S', T \setminus T']$ and $G_B[S \setminus S', T']$ are almost complete. We aim, similarly to the proof of Lemma 29 to find a red cycle C , and two blue paths P_1, P_2 , whose removal from G leaves two balanced bipartite subgraphs of the aforementioned graphs. We then find Hamilton paths in the remainder subgraphs which together with P_1, P_2 form a blue cycle with vertex set $V(G) \setminus V(C)$. We remark that we run into some technical difficulties when X is non-empty.

Proof of Lemma 30. We abuse notation and slightly change the definition of S, T and X as follows. Denote

$$X = \{x, y\} \quad (13)$$

(if $|X| < 2$, add vertices to X arbitrarily). Recall that by the conditions of Lemma 30, we may assume that $\deg_R(x, S) \leq \varepsilon n$ or $\deg_R(x, T) \leq \varepsilon n$. If the former holds, we move x from X to T , otherwise we move x from X to S . Similarly, if $\deg_B(y, S) \geq n/32$ we move y from X to T and otherwise, if $\deg_B(y, T) \geq n/32$, we put y in S . After the modification, we have either $X = \emptyset$, or $X = \{y\}$ and $\deg_B(y) \leq n/16$. Furthermore, the only red edges in $G[S, T]$ are adjacent to x or y .

We may assume that there exist subsets $S' \subseteq S, T' \subseteq T$ of size at least $(1/4 - 100\varepsilon)n$ such that $e(G[S', T']) \leq 25\varepsilon n^2$, because otherwise the proof can be completed immediately by Lemma 29. It is easy to deduce the following claim (we omit the exact details of the proof).

Claim 56. *The following holds for some $\eta = \eta(\varepsilon) \geq \varepsilon$. There exist partitions $\{S_1, S_2\}$ of S and $\{T_1, T_2\}$ of T with the following properties.*

- $|S_1|, |S_2|, |T_1|, |T_2| \geq (1/4 - \eta)n$.
- All but at most ηn vertices of $S_2 \cup T_1$ have degree at most ηn in $G_B[S_2, T_1]$.
- The graphs $G_B[S_i, T_i]$ have minimum degree at least $n/64$. Furthermore, all but at most ηn vertices in these graphs have degree at least $(1/4 - \eta)n$.
- All but at most ηn vertices in the graphs $G[S_1, S_2]$ and $G[T_1, T_2]$ have degree at least $(1/4 - \eta)n$.

We shall also use the following claim which may be easily verified by Theorem 19.

Claim 57. *Let $S' \subseteq S_i$ and $T' \subseteq T_i$ be sets of equal size such that $|(S_i \cup T_i) \setminus (S' \cup T')| \leq 10\eta n$. Then $H = G_B[S', T']$ is Hamiltonian. Furthermore, for every $s \in S', t \in T'$, H contains a Hamilton path with ends s and t .*

Without loss of generality, we assume $|S| \geq |T|$. Denote $k = |S| - |T|$. We consider two cases according to the size of X .

Case 1: $X = \emptyset$

Let A be the set of red neighbours of x in $G[S, T]$ and B the set of red neighbours of y in $G[S, T]$. Then

$$|A| \leq n/16 \text{ and } |B| \leq (1/2 - 1/64)n. \quad (14)$$

If $k \leq 1$, we may find a partition of $V(G)$ into a blue cycle and a red one as follows. Pick any red cycle C_1 in S of length k , so C_1 is either the empty set or a vertex. Denote $S' = S \setminus V(C_1)$ and consider the balanced bipartite graph $H = G_B[S', T]$. Then

$$\deg_H(u) \geq \begin{cases} n/64 & u \in \{x, y\} \\ (n - k)/2 - (n/4 - 1) - 2 = n/4 - k/2 - 1 & u \in A \\ n/4 - k/2 & u \in B \setminus A \\ n/4 - k/2 + 1 & \text{otherwise} \end{cases}$$

It follows by Corollary 20 that H is Hamiltonian (note that B is a subset of either S or T), implying that the desired partition of $V(G)$ into a red cycle and a blue exists.

Case 1.1: $|T_1| < |S_1|$ and $|T_2| < |S_2|$

Recall that $k \geq 2$, so

$$|S| = (n + k)/2 \geq n/2 + 1. \quad (15)$$

Claim 58. *The graph $G_B[S_1, T_2] \cup G_B[S_2, T_1]$ contains a (blue) matching of size 2.*

Proof. If $|S_2| > n/4 + 1$, then every vertex in T_1 has at least three neighbours in S_2 , thus every vertex in $T_1 \setminus (A \cup \{x, y\})$ has at least two blue neighbours in S_2 , implying that we may find the desired matching. Similarly, if $|S_1| > n/4 + 1$, the graph $G_B[S_1, T_2]$ contains a matching of size 2. Thus we may assume that $|S_1|, |S_2| \leq n/4 + 1$, implying that $|S_1|, |S_2| \geq n/4$.

Without loss of generality, $|S_1| \geq n/4, |S_2| > n/4$. Hence every vertex in $T_1 \setminus (A \cup \{x, y\})$ has at least two neighbours in S_2 . If $y \notin S_2$, these two neighbours are both blue, and we may find the required matching in $G_B[S_2, T_1]$. If $y \in S_2$, we conclude similarly that both $G[S_1, T_2]$ and $G[S_2, T_1]$ contain at least one blue edge, implying that the desired matching exists. \square

We shall assume the following; the other cases can be argued similarly.

$$\{s_1 t_1, s_2 t_2\} \text{ is a blue matching in } G[S_2, T_1], \text{ where } s_1, s_2 \in S_2 \text{ and } t_1, t_2 \in T_1. \quad (16)$$

It is easy to verify that one of the following holds.

1. $G_B[S_1]$ contains a path of length $5\eta n$.
2. $G_B[S_1, T_2]$ contains a path of length $5\eta n$.
3. $G_B[S_1, S_2]$ contains a path of length $5\eta n$.
4. $G_R[S_1]$ contains a cycle of length l for every $l \leq 5\eta n$, and $G_B[S_2]$ contains a path of length $5\eta n$.
5. For every $l_1, l_2 \leq 5\eta n$, $G_R[S_1, S_2]$ contains a cycle with l_1 vertices from S_1 and l_2 vertices from S_1 .

Indeed, if the first three conditions do not hold, by Theorem 21, the graphs $G_R[S_1]$ and $G_R[S_1, S_2]$ are almost complete, so either the third condition holds or $G_R[S]$ is almost complete.

If Condition 1 holds, we conclude that $G_B[S_1 \cup T_1]$ has a Hamilton path Q_1 with ends t_1, t_2 . Indeed, let P_1 be a path in $G_B[S_1]$ on $|S_1| - |T_1| + 2$ vertices (note that this quantity is smaller than $5\eta n$). By Claim 57, we may extend P_1 to a Hamilton path of $G_B[S_1 \cup T_1]$ with ends t_1 and t_2 . Now consider $G[S_2]$. Note that $G[S_2]$ is almost complete (as $G[S_2, T_1]$ is almost empty), thus it either contains a blue path of length $|S_2| - |T_2| - 1$ or a red cycle of length $|S_2| - |T_2| - 1$. In any case, we may partition $S_2 \cup T_2$ into a red cycle C and a blue path Q_2 with ends s_1, s_2 (where in the former

case the red cycle is empty). By joining Q_1 and Q_2 using the edges s_1t_1 and s_2t_2 , we obtain a blue cycle on the vertices $V(G) \setminus V(C)$.

If Condition 2 holds, let P_1 be a path in $G_B[S_1, T_2]$ with ends $s \in S_1, t \in T_2$ with exactly $|S_1| - |T_1| + 1$ vertices from S_1 . Denote by U the set of interior vertices of P_1 and let $S'_1 = S_1 \setminus U$ and $T'_2 = T_2 \setminus U$. By Claim 57 the graph $G_B[S'_1, T_1]$ has a Hamilton path Q_1 with ends s, t_1 . Consider the graph $G[S_2 \cup T'_2]$. We conclude as before that $G[S_1, T'_1]$ can be partitioned into a blue path with ends s_1, t and a red cycle, completing the required partition of $V(G)$ into a blue cycle and a red one.

We now assume that Condition 2 does not hold, so $G[S_1, T_2]$ is almost empty and $G[T_2]$ is almost complete. Suppose that Condition 3 holds. Let P_1 be a path in $G_B[S_1, S_2]$ with ends $s_3 \in S_1, s_4 \in S_2$ and exactly $|S_1| - |T_1| + 1$ vertices from S_1 . Define U to be the set of interior vertices of P_1 and let $S'_i = S_i \setminus U$. It can be shown as before that $G_B[S'_1, T_1]$ has a Hamilton path with ends s_3, t_1 and that $G_B[S'_2, T_2]$ can be partitioned into a blue path with ends s_1, s_4 and a red cycle. Note that it may happen that $|T_2| \geq |S'_2|$ in which case the red cycle is contained in T_2 (here we use the assumption that $G[T_2]$ is almost complete).

Now suppose that Condition 4 holds. It follows that $G[S_1 \cup T_1]$ may be partitioned into a blue path with ends t_1, t_2 and a red cycle, and that $G[S_2 \cup T_2]$ contains a blue Hamilton path with ends s_1, s_2 .

Finally, if Condition 5 holds, let C be a red cycle consisting of $|S_1| - |T_1| + 1$ vertices from S_1 and $|S_2| - |T_2| - 1$ vertices from S_2 . As before, the graphs $G[S_1 \setminus V(C), T_1]$ and $G[S_2 \setminus V(C), T_2]$ have blue Hamilton paths with ends t_1, t_2 and s_1, s_2 respectively.

Case 1.2: $|S_1| = |T_1|$ or $|S_2| = |T_2|$

Suppose that $|S_1| = |T_1|$. Similarly to Claim 58, we claim that $G[S_2, T_1]$ contains a blue matching of size 2. Indeed, if either $|S_2| > n/4 + 1$ or $|T_1| = |S_1| > n/4 + 1$ we may find the required matching as in the proof of the claim. Otherwise, we have $n/4 \leq |S_2|, |S_1| \leq n/4 + 1$ (because $|S| \geq n/2 + 1$). If $|S_2| > n/4$ and $y \notin S_2$, every vertex in $T_1 \setminus (A \cup \{x, y\})$ has two blue neighbours in S_2 , and the required matching can be found. A similar argument holds if $|T_1| > n/4$ and $y \notin T_1$. So, we may assume that $|T_1| = n/4, |S_2| = n/4 + 1$ and $y \in S_2$, or $|S_2| = n/4, |T_1| = n/4 + 1$ and $y \in T_1$. In the former case every vertex in $S_2 \setminus (A \cup \{x, y\})$ has a blue neighbour in T_1 , and similarly every vertex in $T_1 \setminus (A \cup \{x\})$ has a blue neighbour in S_2 , which implies the existence of the desired matching. The latter case can be argued similarly. Given this matching, we can proceed to find a partition of the vertices into a red cycle and a blue one as in Case 1.1.

Similarly, if $|S_2| = |T_2|$, it follows that $G[S_1, T_2]$ contains a blue matching of size 2 and we proceed as before.

Case 1.3: $|S_1| < |T_1|$

In order to obtain the required balanced subgraph of $G[S_1, T_1]$, we use the following claim.

Claim 59. *There exist vertex-disjoint paths Q_1, Q_2 in $G_B[S, T]$ satisfying the following properties.*

- $|Q_1|, |Q_2| \leq 32\eta n$.
- Q_i has one end in S_2 and the other in T_1 .
- Denote $U = V(Q_1) \cup V(Q_2)$. Then

$$|U \cap T_1| - |U \cap S_1| = |T_1| - |S_1| + 1.$$

Proof. Consider the bipartite graph $H = G_B[T_1, S']$ where $S' = S \setminus \{u\}$ for some fixed $u \in S_1$. Recall that $|S'| = |S| - 1 \geq n/2$ by (15). We show that there exist two edge-disjoint matchings M_1, M_2 of size $|T_1| - |S_1| + 1$ in $G[S_2, T_1]$ whose union contains no cycles. To that end, we show first that H contains a matching saturating T_1 , by showing that H satisfies Hall's condition, namely that for every $W \subseteq T_1$, we have $|N_H(W)| \geq |W|$.

Recall that x, y are the vertices that were in X originally (13), and A, B are their red neighbourhoods in $G[S, T]$. We consider four ranges for the size of W .

- $|W| \leq 2$. Here Hall's condition holds trivially because the minimum degree of a vertex from T_1 is larger than 2.
- $3 \leq |W| \leq n/4 - 1$. Recall that every vertex in T_1 , except for possibly x and y , has degree at least $n/2 - (n/4 + 1) = n/4 - 1$ in H . Thus, in this case we have

$$|N_H(W)| \geq n/4 - 1 \geq |W|.$$

- $n/4 - 1 < |W| \leq n/4$. By the lower bound on the size of A , there exists $w \in W \setminus (A \cup \{x, y\})$. w has at most one red neighbour in S' (namely y), thus

$$|N_H(W)| \geq \deg_B(w, S') \geq n/2 - n/4 = n/4 \geq |W|.$$

- $|W| > n/4$. In this case, every vertex in $S' \setminus (A \cup \{x, y\})$ has a neighbour in W , so

$$|N_H(W)| \geq |S'| - (n/16 + 2) \geq |T_1| \geq |W|.$$

It follows that there is a matching in $G_B[S', T_1]$ that saturates T_1 . We take M_1 to be a sub-matching in $G_B[T_1, S_2]$ of size $|T_1| - |S_1| + 1 \leq 5\eta n$ (this is possible because at most $|S_1| - 1$ of the edges are incident with S_1).

Consider the graph H' obtained from H by removing the edges of H spanned by $V(M_1)$. It is easy to check by a similar analysis that H' contains a matching saturating T_1 . Indeed, if $1 \leq |W| \leq 5\eta n + 2$, we clearly have $|N_{H'}(W)| \geq |W|$. If $5\eta n + 2 < |W| \leq n/4 - 1$, we have $|N_{H'}(W)| \geq n/4 - 1 \geq |W|$. If $n/4 - 1 < |W| \leq n/4$, we may pick a vertex $w \in W \setminus (A \cup V(M_1) \cup \{x, y\})$ and continue as

above. Finally, if $|W| > n/4$, every vertex in $S' \setminus (V(M_1) \cup A \cup \{x, y\})$ has a blue neighbour in W , so $|N_H(W)| > |W|$. So there exists a matching M_2 in $G_B[T_1, S_2]$, edge-disjoint of M_1 , of size $|T_1| - |S_1| + 1$.

Let F be the graph $(V(G), E(M_1) \cup E(M_2))$. Note that by the choice of H' , F contains is a path forest that consists of paths of length 1 or 2. We modify F as follows. If F contains at least one component that is a single edge, then remove any two edges from F that do not belong to this component; otherwise, F consists only of 2-paths, pick any two such paths and remove one edge from each. Denote the modified graph by F' , and let r_i be the number of vertices in F' with degree i in T_1 . We have

$$|T_1| - |S_1| = |E(F')|/2 = r_2 + r_1/2. \quad (17)$$

We now obtain paths Q_1, Q_2 as follows. We start with the collection of paths in F' and keep modifying it, as follows.

1. As long as there is a path P_1 with both ends in T_1 , pick a path P_2 with one end in S_2 and the other in T_1 (we make sure that there is always such a path). Connect P_1 and P_2 by a path in $G_B[S_1, T_1]$ of length at most 4, which is vertex-disjoint of all other paths, and replace P_1 and P_2 with the extended path (note that this extended path has one end in S_2 and one in T_1).
2. Once there are no paths with both ends in T_1 , but there is a path with both ends in S_2 , replace it and some path with one end in S_2 and one in T_1 with an extended one, obtained by connecting the two using a path of length at most 4 in $G_B[S_2, T_2]$.
3. Once there are only paths with one end in S_2 and one in T_1 , pick three such paths (if possible). Replace them with an extended path, obtained by connecting them with paths in $G_B[S_1, T_1]$ and $G_B[S_2, T_2]$ of length at most 4.

We note that we start the process with an even (and positive) number of paths with one end in S_2 and the other in T_1 , and this number remains unchanged by Stages 1 and 2, and drops by exactly 2 in each step of Stage 3. Hence we eventually reach a point when there are exactly two paths, which we denote Q_1, Q_2 , and both have one end in S_2 and one in T_1 . We claim that Q_1, Q_2 satisfy the requirements of Claim 59. Indeed, the first property follows because the total number of edges is at most $4|E(F')| = 8(|T_1| - |S_1|) \leq 32\eta n$. The second property follows by choice of Q_1, Q_2 . It remains to show that the last property holds. One can check that

$$\begin{aligned} |U \cap T_1| - |U \cap S_1| &= r_2 + (r_1 - 2)/2 + 2 = (2r_2 + r_1)/2 + 1 \\ &= |E(F')|/2 + 1 = |T_1| - |S_1| + 1, \end{aligned}$$

as required. □

Let Q_1, Q_2 be paths as in Claim 59. Denote the ends of Q_i by $s_i \in S_2, t_i \in T_1$, let U' be the set of interior vertices of these paths, and let $S'_i = S_i \setminus U'$ and $T'_i = T_i \setminus U'$. It is easy to verify that

$G_B[S'_1, T'_1]$ has a Hamilton path with ends t_1, t_2 and that $G[S'_2 \cup T'_2]$ may be partitioned into a blue path with ends s_1, s_2 and a red cycle (note that $|S'_2| \geq |T'_2|$, and we use the fact that $G[S_2]$ is almost complete).

Case 1.4: $|S_2| < |T_2|$

Here we have $|S_1| \geq |T_1| + 2$. If $G[S_1, T_2]$ contains a blue path of length $10\eta n$, we may proceed as in Condition 2 from Case 1.1. Otherwise, $G[S_1, T_2]$ has few edges, so $G[S_1]$ contains many edges and we may proceed as in the previous case.

Case 2: $X = \{y\}$

We start by showing that if $k = |S| - |T| \leq 2$, we may partition $V(G)$ into a red cycle and a blue one. Consider $A = N_R(y, S)$ and recall that $|A| \geq n/8$. If $k = 0$, we pick the red cycle C_1 to be (y) . If $k = 1$, we pick the red cycle C_1 to be a red edge (yz) , where $z \in A$. If $k = 2$ and A contains a red edge uv , we pick the red cycle C_1 to be the triangle (uvy) .

In each of these cases, the remainder graph $H = G_B[S', T]$, where $S' = S \setminus V(C_1)$ is a balanced bipartite graph. To obtain the desired partition of $V(G)$ into a red cycle and a blue one, we show that H is Hamiltonian. Indeed, let B be the set of red neighbours of x in $G[S', T]$ (so $|B| \leq n/16$). Then

$$\deg_H(u) \geq \begin{cases} (n - k - 1)/2 - (n/4 - 1) - n/16 = 3n/16 + 1/2 - k/2 & v = x \\ (n - k - 1)/2 - (n/4 - 1) - 1 = n/4 - k/2 - 1/2 & v \in B \\ (n - k - 1)/2 - (n/4 - 1) = n/4 - k/2 + 1/2 & \text{otherwise} \end{cases}$$

It follows from Corollary 20 that indeed, H is Hamiltonian.

It remains to consider the case where $k = 2$ and A contains no red edges. Suppose first that y has two blue neighbours in S . Consider the graph $H = G_B[S', T']$, where $S' = S \setminus \{u\}$ for some $u \in S$ and $T' = T \cup \{y\}$. We claim that H is Hamiltonian. The desired partition may thus be obtained by letting (u) be the red cycle. Indeed, let B be the set of red neighbours of x in $G[S', T']$ (so $|B| \leq n/16$). Then

$$\deg_H(v) \geq \begin{cases} 2 & v = y \\ (n - 1)/2 - n/4 - n/16 = 3n/16 - 1/2 & v = x \\ (n - 1)/2 - (n/4 + 1) = n/4 - 3/2 & v \in B \\ (n - 1)/2 - n/4 = n/4 - 1/2 & v \in S' \setminus (B \cup \{x, y\}) \\ (n - 1)/2 - (n/4 - 1) = n/4 + 1/2 & v \in T' \setminus (B \cup \{x, y\}) \end{cases}$$

By Corollary 20, H is indeed Hamiltonian.

We may now assume that y has at most one blue neighbour in S . It follows that $\deg_R(y, S) = |A| \geq |S| - n/4 = n/4 + 1/2$. Hence, $G[A]$ contains an edge uv . Recall that we assumed that A contains

no red edges, thus uv is a blue edge. Let $w \in A \setminus \{u, v\}$. Consider the graph $H = G_B[S', T]$ where $S' = S \setminus \{w\}$. We claim that H contains a blue Hamilton path with ends u, v – this would imply that $H \setminus \{y\}$ has a Hamilton blue cycle. Indeed, let H' be the graph obtained from H by adding a vertex z to T which is connected only to u, v . Clearly, H contains a Hamilton path with ends u, v if and only if H' is Hamiltonian. As in the previous argument, it follows from Theorem 19 that H' is Hamiltonian.

From now on, we assume that $k \geq 3$, so $|S| \geq n/2 + 1$. Denote

$$A_i = N_R(y, S_i) \text{ and } B_i = N_R(y, T_i). \quad (18)$$

Recall that $\deg_B(y) \leq n/16$. It follows that at most one of the sets A_1, A_2, B_1, B_2 has size at most $n/16$.

Case 2.1: $e(G_B[S_1, T_2]) \geq 10\eta n^2$

Suppose first that $|A_2| \geq n/16$. This case can be treated similarly to previous cases where $X = \emptyset$. If $|S_1| > |T_1|$, we may continue similarly to Condition 2 from Case 1.1. If $|S_1| = |T_1|$ we continue similarly to Case 1.2 and if $|S_1| < |T_1|$, we continue as in Case 1.3. The only difference in the arguments is that when considering the graph $G[S'_2 \cup T'_2]$ we partition it into a blue path with suitable ends and a red path contained in $A_2 = N_R(y, S_2)$ rather than a red cycle (by the lower bound on A_2 , we have that $G[A_2]$ is almost complete, hence we may indeed do so). We now suppose that $|A_2| \leq n/16$, so $|A_1|, |B_1|, |B_2| \geq n/16$.

Suppose next that $G_B[S_1, S_2]$ contains a path of length $10\eta n$. We may pick vertex-disjoint paths $P_1 \subseteq G_B[S_1, S_2]$ and $P_2 \subseteq G_B[S_1, T_2]$ of length at most $10\eta n$ such that the following assertions hold.

- P_1 has ends $s_1 \in S_1, s_2 \in S_2$ and P_2 has ends $s_3 \in S_1, t_3 \in T_2$.
- $|S_i \setminus V(P_1)| < |T_i|$ for $i \in [2]$.
- Denote by U the set of the interior vertices of the paths P_1, P_2 and $S'_i = S_i \setminus U, T'_i = T_i \setminus U$. Then $|S'_2| = |T'_2|$ (and thus $|S'_1| < |T'_1|$).

Indeed, to construct the paths P_1, P_2 , first pick $P_1 \in G_B[S_1, S_2]$ to be long enough so that the second condition holds. Then by Theorem 21, the graph $G[S_1, T_2] \setminus V(P_1)$ contains a path of length $10\eta n$, and we may take P_2 to be a subpath satisfying the third condition. We continue as above, by finding a Hamilton path in $G_B[S'_2, T'_2]$ with ends s_2, t_3 and partitioning $G[S'_1, T'_1]$ into a blue path with ends s_1, s_3 and a red path contained in B_1 .

We may now assume that $G_B[S_1, S_2]$ does not have a path of length $10\eta n$, so $G_R[S_1, S_2]$ is almost complete. It follows that we may pick $u, v \in A_1$ such that the set $D = N_R(u, S_2) \cap N_R(v, S_2)$ has

size at least $n/8$. Define $S'_1 = S_1 \setminus \{u, v\}$ and $S' = S \setminus \{u, v\}$. We shall now continue as before, partitioning the graph $G[S' \cup T]$ into a blue cycle and a red path with ends in D .

If $|S'_1| > |T_1|$, we continue as in condition 2 of Case 1.1, but when we consider the remainder of the graph $G[S_2 \cup T_2]$, we partition it into a blue path with suitable ends and a red path contained in D . Note that in order for this path to complete the path (uyv) into a cycle, it has to contain at least one vertex. Recall that $|S| \geq |T| + 3$ so this is indeed possible.

If $|S'_1| = |T_1|$, we note that there are blue edges in $G_B[S'_1, T_2]$ and $G_B[S_2, T_1]$. We pick one edge from each of these graphs, extend them by connecting them to a Hamilton path in $G_B[S'_1, T_1]$ and proceed as usual to find a partition of $G[S_2, T_2]$ to a blue path with suitable ends and a non-empty path in D .

Finally, we need to deal with the case $|S'_1| < |T_1|$. Here we can follow the argument of Case 1.3. Note that we need to ensure that a version of Claim 59 holds for the graph $G[S', T]$. Indeed, take $H = G_B[S'', T_1]$, where S'' is obtained from S' by removing a vertex from S'_1 . We claim that H has a matching saturating T_1 . Indeed, let $W \subseteq T_1$. We consider four ranges for the size of W .

- $|W| = 1$. Clearly, $|N_H(W)| \geq |W|$.
- $2 \leq |W| \leq n/4 - 2$. Since any vertex of W other than x has at most one red neighbour in $G[S'', T]$, we have

$$|N_H(W)| \geq |S''| - n/4 = |S| - n/4 - 3 \geq n/4 - 2 \geq |W|.$$

- $n/4 - 2 < |W| \leq n/4 - 1$. There is a vertex in W which has no red neighbours in $G[S'', T]$, thus $|N_H(W)| \geq n/4 - 1 \geq |W|$.
- $|W| > n/4 - 1$. Many vertices of S'' have at least one neighbour in W , implying that $|N_H(W)| > |T_1| \geq |W|$.

Thus we may pick a matching M_1 from in $G_B[S_2, T_1]$ of size $|T_1| - |S'_1| + 1$. Repeating almost the same argument, we deduce that the graph H' obtained from H by removing all edges spanned by $V(M)$, has a matching saturating T_1 . From here we may continue as in Claim 59 to obtain paths Q_1, Q_2 . Finally, as explained before, when considering the graph remaining from $G[S_2, T_2]$, we partition it into a blue path with suitable ends and a red path, on at least one vertex, contained in D .

From now on we may assume that $e(G_B[S_1, T_2]) \leq 10\eta n^2$, so S_1, S_2 become practically interchangeable, and all of $G[S_i], G[T_i]$ are almost complete.

Case 2.2: $G_B[S_1, S_2]$ has a path of length $20\eta n$

If $|S_1| > |T_1|$ and $|A_2|, |B_2| \geq n/16$, we continue as in Condition 3 in Case 1.1, but when considering the graph remaining from $G[S_2 \cup T_2]$ we partition it into a blue path with the given ends and a red

path (which may be empty) contained in either A_2 or B_2 . If $|S_1| = |T_1|$ and $|A_2|, |B_2| \geq n/16$, it is easy to check that $G_B[S_2, T_1]$ is non empty, so we may proceed as before. The case $|S_2| \geq |T_2|$ and $|A_1|, |B_1| \geq n/16$ follows analogously.

Since at most one of the sets A_1, A_2, B_1, B_2 has size at most $n/16$, it remains to consider the case where $|S_1| < |T_1|$ or $|S_2| < |T_2|$. Without loss of generality, $|S_1| < |T_1|$, so $|S_2| > |T_2|$. If $|A_1|, |B_1| \geq n/16$, we are done. Thus we may assume $|A_2| \geq n/16$. But then we may proceed as in Case 1.3, to partition $V(G)$ into a blue cycle and a red path contained in A_2 .

From now on, we may assume that $G_B[S_1, S_2]$ has no path of length $20\eta n$, implying that $G_R[S_1, S_2]$ is almost complete. Also, without loss of generality, $|A_2| \geq n/16$.

Case 2.3: $|S_1| \geq |T_1| + 1$ and $|S_2| \geq |T_2| + 3$

It is easy to check that in this case $G[S_1, T_2] \cup G[S_2, T_1]$ contains a blue matching of size 2. Denote the matching by $\{e_1, e_2\}$ and without loss of generality suppose e_i has ends $s_i \in S_2, t_i \in T_1$. If $G[S_1]$ has a blue path of length $20\eta n$, the graph $G_B[S_1 \cup T_1]$ has a Hamilton path with ends t_1, t_2 . We proceed as before, to partition $G[S_2 \cup T_2]$ into a blue path with ends s_1, s_2 and a red path contained in A_2 .

We may thus assume that $G_R[S_1]$ is almost complete. If $e(G_B[S_2]) \geq 10\eta n^2$, pick $u, v \in A_2$ such that the set $D = N_R(u, S_1) \cap N_R(v, S_1)$ has size at least $n/8$. Denote $S'_2 = S_2 \setminus \{u, v\}$. It is easy to verify that $G_B[S'_2, T_2]$ has a Hamilton path with ends s_1, s_2 . We form a red cycle by picking a red path P_2 in D of length $|S_1| - |T_1|$ and joining it to $(u y v)$ (note that this is indeed a cycle since $|S_1| - |T_1| \geq 1$). The graph $G_B[S_1 \setminus V(P_2), T_1]$ has a Hamilton path with ends t_1, t_2 , completing the required partition.

Finally, if $e(G_B[S_2]) \leq 10\eta n^2$, we have that $G_R[S]$ is almost complete, hence we may pick a red cycle C in $S \cup \{y\}$ with the following properties.

- $y \in V(C)$.
- $|V(C) \cap S_1| = |S_1| - |T_1| + 1$.
- $|V(C) \cap S_2| = |S_2| - |T_2| - 1$.

It is easy to verify that the graph $G_B \setminus V(C)$ is Hamiltonian.

Case 2.4: $|S_1| = |T_1|, |S_2| = |T_2| + 2$ or $|S_2| = |T_2| + 1$

These cases may be dealt with similarly to the previous one. Denote $H_1 = G_B[S_1, T_2]$, $H_2 = G_B[S_2, T_1]$ and $H = H_1 \cup H_2$.

If $|S_1| = |T_1|$, if there is a 2-matching in H with at least one edge from H_2 , we may continue as in the previous case. Namely, we need to show that H_2 is non-empty. This follows if $|S_2| > n/4 - 1$

or $|T_1| > n/4 - 1$, so we may assume that $|S_2|, |T_1| \leq n/4 - 1$, implying that $|S| = |S_1| + |S_2| = |T_1| + |S_2| \leq n/2 - 2$, a contradiction.

If $|S_2| = |T_2| + 2$ (so $|S_1| \geq |T_1| + 1$) we need to show that H_1 is non-empty. If it is empty, we have $|S_1|, |T_2| \leq n/4 - 1$, implying that $|S| \leq n/2$, a contradiction.

If $|S_2| = |T_2| + 1$, we need to show that H_1 contains a 2-matching. If not, we have $|S_1|, |T_2| \leq n/4$, and in addition either $|S_1| \leq n/4 - 1$ or $|T_2| \leq n/4 - 1$. In particular, $|S_1| + |T_2| \leq n/2 - 1$, hence $|S| \leq n/2$, a contradiction.

Case 2.5: $|S_1| < |T_1|$

We proceed as in Case 1.3, partitioning the graph remaining from $G[S_2 \cup T_2]$ into a blue path with suitable ends and a red path contained in A_2 .

Case 2.6: $|S_2| \leq |T_2|$

We may continue as in the last part of Case 2.1. We pick $u, v \in A_2$ such that $|N_R(u, v) \cap S_1| \geq n/8$ and proceed to partition $G[S' \cup T]$ (where $S' = S \setminus \{u, v\}$) into a blue cycle and a red path contained in the red neighbourhood of u, v in S_1 , using an analogue of Claim 59. \square

The proof of Lemma 30 concludes the proof of our main Theorem, Theorem 1. We finish this paper with some concluding remarks.

16 Concluding Remarks

As a further line of research, one may consider colourings of K_n with more than two colours. Gyárfás [13] conjectured that for every r -colouring of K_n , the vertex set may be partitioned into at most r monochromatic paths. Erdős, Gyárfás and Pyber [10] considered partitions into monochromatic cycles rather than paths. They defined $c(r)$ to be the smallest t such that whenever a complete graph is r -coloured, it may be partitioned into t monochromatic cycles. They proved that $c(r)$ is bounded, and furthermore $c(r) \leq cr^2 \log r$ for some constant c . Note that Bessy and Thomassé's result [5], mentioned in the introduction, implies that $c(2) = 2$. Erdős, Gyárfás and Pyber [10] conjectured that $c(r) = r$ for all r .

Gyárfás, Ruszinkó, Sárközy and Szemerédi [14] proved that $c(r) \leq cr \log r$, which is the best known upper bound on $c(r)$ so far. The same authors [15] proved an approximate result of the last conjecture for $r = 3$. Furthermore, they showed that for large enough n , if K_n is 3-coloured, it may be partitioned into 17 monochromatic cycles. However, it turns out that the full conjecture is false, even for $r = 3$, as shown by Pokrovskiy [24]. Nevertheless, in the same paper, he proved Gyárfás's conjecture for $r = 3$, so it may still be the case that this conjecture holds in general. In

addition, the counter examples given in [24] are 3-colourings of K_n for which all but one vertex may be covered by vertex-disjoint monochromatic paths. This raises the following question: is it true that for every r -colouring of K_n all but at most $c = c(r)$ vertices may be covered by r vertex-disjoint monochromatic paths? In a forthcoming paper [19], we use methods introduced in this paper to answer this in the affirmative for $r = 3$, with c being about 60. Pokrovskiy [23] recently proved the same result, using different methods, with $c = 43000$.

Finally, it is natural to consider the Schelp-type version of these problems, namely for graphs with large minimum degree rather than for complete graph. An example for a concrete question of this type is: what is the smallest value of c such that any 3-coloured graph G on n vertices and minimum degree $\delta(G) \geq cn$ can be partitioned into three monochromatic paths?

We believe that the methods we have employed for the proof of Theorem 1 may prove useful in resolving the latter question, as well as many others, regarding partitions of r -coloured graphs into paths and cycles.

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