Fractional triangle decompositions in almost complete graphs

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Abstract

We prove that every n-vertex graph with at least $\binom{n}{2} - (n-4)$ edges has a fractional triangle decomposition, for $n \geq 7$. This is a key ingredient in our proof, given in a companion paper, that every n-vertex 2-coloured complete graph contains $n^2/12 + o(n^2)$ edge-disjoint monochromatic triangles, which confirms a conjecture of Erdős.

1 Introduction

A *triangle packing* in a graph G is a collection of edge-disjoint triangles, and a *triangle decomposition* is a triangle packing that covers all the edges. A *fractional triangle packing* in a graph G is an assignment of weights in $[0, 1]$ to the triangles in G, such that the total weight of every edge is at most 1; namely, $\sum_{w \in V(G)} \omega(uvw) \leq 1$ for every edge uv (where $\omega(uvw) = 0$ if uvw is not a triangle). Given a triangle packing ω and an edge $e = uv$ in G, we define $\omega(e) = \sum_{w \in V(G)} \omega(uvw)$; so $\omega(e) \leq 1$. A *fractional triangle decomposition* in a graph G is a fractional triangle packing ω, satisfying that $\omega(e) = 1$ for every edge e. Our main result in this paper is the following theorem, which shows that almost complete graphs have fractional triangle decompositions.

Theorem 1.1. *Let G be a graph on $n \geq 7$ vertices with $e(G) \geq \binom{n}{2} - (n-4)$. Then there is a fractional triangle decomposition in G.*

Theorem 1.1 is tight in two ways: the complete graph on six vertices with two edges removed (intersecting or not) does not have a fractional triangle decomposition; and the graph on vertex set $[n]$ with non-edges $\{xn : x \in \{4, \ldots, n-1\}\} \cup \{12\}$ is an n-vertex graph with $n-3$ non-edges that does not have a fractional triangle decomposition.

Our main motivation for proving Theorem 1.1 is our [8] proof that every n-vertex 2-coloured complete graph has $n^2/12 + o(n^2)$ edge-disjoint monochromatic triangles, which confirms a conjecture of Erdős [5]. To prove the conjecture, we use a reduction to fractional monochromatic triangle

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packings, due to Haxell and Rödl [10]. Our proof there is inductive, and Theorem 1.1 is a key ingredient in the induction step.

A well-known conjecture of Nash-Williams [12] asserts that every n-vertex graph G with minimum degree at least $3n/4$, where n is large and G satisfies certain 'divisibility conditions', has a triangle decomposition. While this conjecture is still open, significant progress towards it has been made. Recently, Delcourt and Postle [2] showed that every n-vertex graph with minimum degree at least $0.83n$ has a fractional triangle decomposition, improving on several previous results (see, e.g., [3, 4, 7, 9, 14]). Combined with a result of Barber, Kühn, Lo and Osthus [1], it follows that the statement obtained by replacing $3/4$ by $0.831n$ in Nash-Williams's conjecture is true. Delcourt and Postle's result (or any result about fractional triangle decompositions in graphs with large minimum degree) can be used to prove Theorem 1.1 for sufficiently large n. However, crucially, in [8] we need Theorem 1.1 to hold for all $n \geq 7$, and we thus prove Theorem 1.1 without relying on such results.

In fact, in [8] we use the following stronger version of Theorem 1.1.

Corollary 1.2. *Let G be a complete graph on $n \geq 7$ vertices, and let $\phi : E(G) \to [0,1]$ be such that $\sum_{e \in E(G)} \phi(e) \geq \binom{n}{2} - (n-4)$. Then there is a fractional triangle packing ω in G such that $\omega(e) = \phi(e)$ for every $e \in E(G)$.*

In order to prove Theorem 1.1, we prove a stronger statement (see Theorem 2.1) by induction, constructing a suitable fractional triangle packing in G using fractional triangle packings of certain graphs related to G on $n-1$ and $n-2$ vertices. The induction base is proved by computer search. Corollary 1.2 follows from Theorem 1.1 via a reduction from weighted graphs to simple graphs (see Lemma 2.4).

Organisation of the paper

In Section 2 we introduces some notation, mention a few preliminaries, and state Theorem 2.1 – a strengthening of Theorem 1.1 which is more amenable to an inductive proof. In Section 3 we prove Lemma 2.4, which will allow us to prove Corollary 1.2 and will be handy for the proof of Theorem 2.1. In Section 4 we describe the algorithm used in our computer search, and explain how it proves Theorems 1.1 and 2.1 for small values of n. Finally, in Section 5 we complete the proof of Theorem 2.1.

2 Preliminaries

Recall that a *fractional triangle packing* in G is an assignment ω of weights in $[0,1]$ to the triangles in G, such that the total weight on every edge of G is at most 1; i.e. $\sum_{w \in V(G)} \omega(uvw) \leq 1$ for every edge uv in G (where $w(uvw) = 0$ whenever uvw is not a triangle). For every edge uv we define $\omega(uv) := \sum_{w \in V(G)} \omega(uvw)$. A *fractional triangle decomposition* in a graph G is a fractional triangle packing ω satisfying $\omega(e) = 1$ for every edge e in G.

The *uncovered weight* in ω is the total uncovered edge-weight, namely $\sum_{e\in E(G)}(1-\omega(e))$. (So a fractional triangle packing ω is a fractional triangle decomposition if and only if the uncovered weight in ω is 0.) Given a graph G, the number of *missing edges* in G is the number of pairs of vertices that are not edges of G.

In order to prove our main result, Theorem 1.1, we prove the following stronger result. We note that the lower bound on n is tight, as there exists a graph on 10 vertices with 10 missing edges, that does not have a fractional triangle packing with uncovered weight at most 4.

Theorem 2.1. *Let G be a graph on n vertices with at most $n-4+a$ missing edges, where $n \geq 11$ and $0 \leq a \leq 4$. Then there is a fractional triangle packing in G with total uncovered weight at most a, such that every triangle has weight at most $1/2$.*

We prove Theorems 1.1 and 2.1 for $n \leq 13$ by computer. More precisely, we prove the following lemma. For a description of our algorithm, and a proof of this lemma using the outcome of the computer search, see Section 4. The certificates relevant to the computer search can be found here.

Lemma 2.2. *Let G be an n-vertex graph with $n-4+a$ missing edges, where $a \in \{0,\dots,4\}$.*

- *If $n \in \{11,12,13\}$ and $a \in \{0,\dots,4\}$, then G has a fractional triangle packing with uncovered weight at most a, such that every triangle in G has weight at most $1/2$.*
- *If $n \in \{7,\dots,10\}$ and $a = 0$, then G has a fractional triangle decomposition.*

It will be convenient for us to assume that G has *exactly* $n-4+a$ missing edges, for some $a \in \{0,\dots,4\}$. To do so, we use the following reduction. Note that Theorem 1.1 follows directly from Theorem 2.1 and Lemmas 2.2 and 2.3.

Lemma 2.3. *Suppose that every graph on n vertices with exactly $m < \binom{n}{2}$ missing edges has a fractional triangle decomposition, such that every triangle has weight at most $\beta \geq 1/3$. Then the same holds for graphs with* at most *m missing edges.*

Proof. We prove by induction that every n-vertex graph with k missing edges, where $k \leq m$, has a fractional triangle decomposition, such that every triangle has weight at most β. The case $k = m$ holds by assumption. Now suppose that the statement holds for k with $1 \leq k \leq m$. Let G be an n-vertex graph with $k-1$ missing edges. Note that G has a triangle. Indeed, by assumption on k, the graph obtained by removing any edge from G has a fractional triangle decomposition; in particular, it contains a triangle (as G has at least one edge). Let uvw be a triangle in G. By assumption on k, each of the graphs $G \setminus \{uv\}$, $G \setminus \{uw\}$ and $G \setminus \{vw\}$ has a fractional triangle decomposition, such that the weight of each triangle is at most β. Taking the average of these three packings, and additionally assigning weight $1/3$ to uvw, we obtain a triangle decomposition in G with no heavy triangles, as required. $\square$

A *weighted graph* is a pair (G, ϕ) where G is a graph and ϕ is an assignment of weights in $[0, 1]$ to the edges of G. The *missing weight* in (G, ϕ) is $\sum_{e \in V(G)^{(2)}}(1 - \phi(e))$, where $\phi(e) = 0$ if e is not an edge of G. A fractional triangle packing ω in (G, ϕ) is a fractional triangle packing ω in G such that $\omega(e) \leq \phi(e)$ for every edge e in G. The *uncovered weight* in ω is, as above, the total uncovered edge-weight, namely $\sum_{e \in E(G)}(\phi(e) - \omega(e))$.

Our proof of Theorem 2.1 is inductive. When applying the induction step, it will be useful for us to have a version of Theorem 2.1 for weighted graphs. This can be achieved by the following lemma.

Lemma 2.4. *Suppose that every graph on n vertices with at most m missing edges has a fractional triangle packing with total uncovered weight at most a, such that every triangle has weight at most β. Then every* weighted *graph with missing weight at most a, has a fractional triangle packing ω with uncovered weight at most a such that every triangle has weight at most β.*

Note that Corollary 1.2 follows immediately from Theorem 1.1 and Lemma 2.4.

In our proof of Theorem 2.1 we make use of the following corollary of Ore's theorem [13], which asserts that if a graph G on $n \geq 3$ vertices satisfies $d(u) + d(w) \geq n$ for every two non-adjacent vertices u and w, then G has a Hamilton cycle.

Corollary 2.5. *Let G be a graph on $n \geq 3$ vertices with at most $n - 3$ missing edges. Then G has a Hamilton cycle.*

A *heavy triangle* in a fractional triangle packing ω is a triangle T with $\omega(T) > 1/2$. Given a graph G on n vertices and a vertex u in G, we denote the degree of u by $d_G(u)$ and its non-degree (namely, the number of non-edges incident with u) by $\bar{d}_G(u)$; so $\bar{d}_G(u) = n - 1 - d_G(u)$. When G is clear from the context, we omit the subscript G.

3 Fractional triangle packings in weighted graphs

In this section we prove Lemma 2.4, which reduces the problem of finding large fractional packings in weighted graphs, to finding such packings in unweighted graphs.

Proof of Lemma 2.4. Let (G, ϕ) be a weighted graph as in the statement of the claim. Suppose first that $\phi(e)$ is rational for every edge e, and let r be such that $\phi(e)r$ is integer for every edge e. We make use of the following claim.

Claim 3.1. *Suppose that $d_1, \ldots, d_N \in \{0, \ldots, r\}$ satisfy $d_1 + \ldots + d_N \leq r \cdot m$. Then there exist subsets $S_1, \ldots, S_r \subseteq [N]$ of size at most m such that every $i \in [N]$ appears in exactly d_i sets S_j with $j \in [r]$.*

Proof. We prove the claim by induction on r. If $r = 0$, the statement holds trivially. Suppose that $r \geq 1$, and that the statement holds for $r-1$. Without loss of generality, suppose that $d_1 \geq \ldots \geq d_N$. Let S_r be the set of indices $i \in [m]$ with $d_i \geq 1$. Define

$$d_i' = \begin{cases} d_i - 1 & i \in S_r \\ d_i & \text{otherwise.} \end{cases}$$

Note that $d_i' \leq r - 1$ for every $i \in [N]$. Indeed, otherwise, $d_1, \ldots, d_{m+1} \geq r$, contradicting the assumption that $\sum_{i \in [N]} d_i \leq r \cdot m$. Moreover, $\sum_{i \in [N]} d_i' \leq (r-1)m$. Indeed, if $|S_r| = m$ then $\sum_{i \in [N]} d_i' = \sum_{i \in [N]} d_i - m \leq (r-1)m$; and if $|S_r| < m$ then $d_m' = \ldots = d_N' = 0$, so $\sum_{i \in [N]} d_i' = \sum_{i \in [m-1]} d_i' < (r-1)m$. It follows that, by induction on r, there exist sets $S_1, \ldots, S_{r-1} \subseteq [N]$ of size at most m, such that every $i \in [N]$ is in exactly d_i' sets S_j with $j \in [r-1]$. The sets $S_1, \ldots, S_r$ satisfy the requirements for $d_1, \ldots, d_N$. $\square$

Note that $(1 - \phi(e))r \in \{0, \ldots, r\}$ for every edge e, and $\sum_{e \in E(G)} (1 - \phi(e))r \leq r \cdot m$. Thus, by Claim 3.1, there exist sets $S_1, \ldots, S_r \subseteq E(G)$ of size at most m, such that every $e \in E(G)$ is in exactly $(1 - \phi(e))r$ sets S_i. Let G_i be the graph on vertex set $V(G)$ with non-edges S_i. Then G_i is a graph on n vertices with at most m non-edges, so by assumption there is a fractional triangle packing ω_i in G_i with uncovered weight at most a, such that all triangles have weight at most β. Let $\omega = (1/r) \cdot \sum_{i \in [r]} \omega_i$. Then ω is a triangle packing in (G, ϕ) (as $\omega(e) \leq (r - (1 - \phi(e))r)/r = \phi(e)$) with uncovered weight at most a such that all triangles have weight at most β. This concludes the proof in the case where $\phi(e)$ is rational for every $e \in E(G)$.

Now consider the general case, where $\phi(e)$ may be irrational for some edges e. For $\ell \in \mathbb{N}$, let ϕ_ℓ be such that $\phi_\ell(e)$ is rational for every edge e, and $\phi(e) \leq \phi_\ell(e) \leq \min\{1, \phi(e) + 1/\ell\}$. Then by the proof for rational edge-weightings, there is a fractional triangle packing $\omega_\ell(e)$ with the requirements stated in the claim. By taking the limit of taking the limit of a converging subsequence of $(\omega_\ell)_\ell$, we find a fractional triangle packing ω that satisfies the requirements for ϕ. $\square$

4 Computer search

In this section we describe the algorithms that we use to prove Lemma 2.2. The certificates relevant to the computer search can be found here.

We say that a pair (N, M) of integers which is *relevant* if

- either $N \in \{11, 12, 13\}$ and $M = \binom{N}{2} - (N - 4 + a)$ for some $a \in \{0, \ldots, 4\}$,
- or $N \in \{7, \ldots, 10\}$ and $M = \binom{N}{2} - (N - 4)$.

4.1 The algorithm

The algorithm receives a pair of integers (N, M) as input. It then performs the following steps.

1. Generate all sequences of integers $(d_1, \dots, d_N)$ such that
   - $d_1 \geq \dots \geq d_N$,
   - $\sum_{i \in [N]} d_i = 2M$,
   - there exists a graph on N vertices with degree sequence $(d_1, \dots, d_N)$.
2. Form an auxiliary acyclic digraph D as follows.
   - The vertices of D are degree sequences of graphs $\mathbf{d} = (d_1, \dots, d_n)$ such that $d_1 \geq \dots \geq d_n$.
   - The *sinks* (namely the vertices of out-degree 0) are the degree sequences generated in step 1.
   - For every $\mathbf{d} = (d_1, \dots, d_n)$ in F, the collection of in-neighbours is defined as follows.
     (a) If $\mathbf{d}$ is the empty sequence, it has no in-neighbours.
     (b) If $\sum_{i \in [n]} > \frac{1}{2}\binom{n}{2}$, set $\mathbf{d}' = (n-1-d_n, \dots, n-1-d_1)$, and let $\mathbf{d}'$ be the only in-neighbour of $\mathbf{d}$.
     (c) Otherwise, if $d_n \in \{0, 1\}$, let i be the least integer satisfying $d_{i+1} \leq 1$. Let the in-neighbourhood of $\mathbf{d}$ consist of sequences $\mathbf{d}' = (d'_1, \dots, d'_i)$ such that
       - $d'_1 \geq \dots \geq d'_i$,
       - $d'_j \leq d_j$ for every $j \in [i]$,
       - $\sum_{j \in [i]} d_j - \sum_{j \in [i]} d'_j \leq \sum_{j \in \{i+1, \dots, n\}} d_j$,
       - $\mathbf{d}'$ is a degree sequence of a graph.
     (d) If neither of the previous conditions hold, let the in-neighbours of $\mathbf{d}$ be the sequences $\mathbf{d}' = (d'_1, \dots, d'_{n-1})$ satisfying
       - $d'_1 \geq \dots \geq d'_{n-1}$,
       - $d'_j \in \{d_j - 1, d_j\}$ for every $j \in [n-1]$,
       - $\sum_{j \in [n-1]} d_j - \sum_{j \in [n-1]} d'_j = d_1$,
       - $\mathbf{d}'$ is a degree sequence of a graph.
   - In particular, the only *source* (namely vertex of in-degree 0) is the empty sequence.
3. For each $\mathbf{d} \in D$, we generate the collection $\mathcal{G}(\mathbf{d})$ of all graphs with degree sequence $\mathbf{d}$, as follows.
   - For $\mathbf{d}$ being the empty set, we set $\mathcal{G}(\mathbf{d})$ to consist of the empty graph with empty vertex set.
   - Suppose that we have calculated $\mathcal{G}(\mathbf{d})$ for some $\mathbf{d} = (d_1, \dots, d_n) \in V(D)$. Then for every out-neighbour $\mathbf{d}'$ of $\mathbf{d}$, let $\mathcal{G}(\mathbf{d}')$ consist of all graphs G' with degree sequence $\mathbf{d}'$, that can be obtained from some $G \in \mathcal{G}(\mathbf{d})$ by
     - taking the complement of G, if the edge $\mathbf{dd}'$ was formed in step 2(b),
     - adding some new vertices to G and joining each of them with at most one (new or existing) vertex, if $\mathbf{dd}'$ was formed according to step 2(c),

– adding a new vertex to G and joining it to at least two existing vertices, if $\mathbf{dd}'$ was formed according to step 2(d).

4. For each sink $\mathbf{d}$ in D (so $\mathbf{d}$ is a degree sequence of an N-vertex graph with M edges), and for each graph $G \in \mathcal{G}(\mathbf{d})$, run a linear program to minimise the uncovered weight of a fractional triangle packing in G with no heavy triangles (i.e. with no triangles of weights larger than $1/2$).

Outcome. For every relevant (N, M), for sinks in the graph D generated for (N, M), all the graphs in $\mathcal{G}(\mathbf{d})$ have a fractional triangle packing with uncovered weight at most a (where $M = \binom{N}{2} - (N - 4 + a)$) with no heavy triangles.

4.2 Proof of Lemma 2.2

It is now easy to prove Lemma 2.2.

Proof. Fix some (N, M), and let D be the directed graph generated by the algorithm for (N, M). It is easy to see that $\mathcal{G}(\mathbf{d})$ is the collection of all graphs with degree sequence $\mathbf{d}$, for every vertex $\mathbf{d}$ in D. Indeed, this can be done by induction, noting that every graph G with degree sequence $\mathbf{d}$ can be obtained from a graph G' with degree sequence $\mathbf{d}'$ for some in-neighbour $\mathbf{d}'$ of $\mathbf{d}$, as described in step 3. In particular, the union of the families $\mathcal{G}(\mathbf{d})$ over all sinks $\mathbf{d}$ of D is the collection of all graphs on N vertices with M edges, using the fact that the set of sinks is the set of degree sequences of such graphs, by steps 1 and 2. It thus follows from the outcome of the algorithm that every N-vertex graph with M edges has a fractional triangle packing with uncovered weight at most a and no heavy triangles, where $M = \binom{N}{2} - (N - 4 + a)$, and (N, M) is relevant. Lemma 2.2 follows. $\square$

4.3 Remarks

We conclude this section with some remarks regarding the algorithm.

1. In order to determine if a sequence $\mathbf{d} = (d_1, \ldots, d_n)$, where $d_1 \geq \ldots \geq d_n$, is a degree sequence of a graph, we use the well-known criterion due to Erdős and Gallai [6], according to which $\mathbf{d}$ is a degree sequence of a graph if and only if $\sum_{i \in [n]} d_i$ is even, and

$$\sum_{i \in [k]} d_i \leq k(k-1) + \sum_{i \in \{k+1, \ldots, n\}} \min\{k, d_i\}$$

   for every $k \in [n]$.

2. For correctness, it is not necessary to allow for edges of D as in steps 2(b) and 2(c). We do include such edges in D, as this means that we will mostly consider degree sequences of

relatively sparse graphs (after 'taking the complements' of degree sequences corresponding to graphs on N vertices with M edges). Such graphs are likely to have many leaves and isolated vertices, and removing them all at once, rather than one by one, decreases the size of D and thus lets the algorithm to run faster.

3. When forming the collections $\mathcal{G}(\mathbf{d})$ in step 3, we adapt an algorithm of McKay and Piperno [11] to detect whether a newly generated graph is isomorphic to a graph that was generated previously.

4. In principle, the fractional triangle packings found in step 4 may be susceptible to rounding errors. To account for this possibility we find a rational approximation of the packings found, using continuous fractions approximations (while ensuring that the weights are non-negative, and that no edge receives weight larger than 1). In practice, the program did not encounter any issues related to rounding errors. Nevertheless, for correctness, this had to be checked.

5 The proof

In this section we prove Theorem 2.1.

Proof. We prove the result by induction on n. By Lemma 2.3, it suffices to prove the statement for graphs with exactly $n-4+a$ non-edges, where $a \in \{0, \dots, 4\}$. The case $n \in \{11, 12, 13\}$ thus follows from Lemma 2.2. Let $n \geq 14$ and suppose that the statement of Theorem 2.1 holds for $n-1$ and $n-2$. Let G be a graph on $n \geq 14$ vertices with exactly $n-4+a$ non-edges, where $a \in \{0, \dots, 4\}$. We consider four cases: there is a vertex u with $\bar{d}(u) > (n+a)/3$; $m \in \{0, 1, 2, 3\}$; $a < 4$ and $m \geq 4$; and $a = 4$ and $m \geq 4$. The latter two carry the main difficulty of the proof.

5.1 Case 1. There is a vertex u with $\bar{d}(u) > (n+a)/3$

Write $\bar{d} := \bar{d}(u)$. Consider the graph $G[N(u)]$; it has $d := |N(u)| = n - 1 - \bar{d}$ vertices and at most $n - 4 + a - \bar{d} = d - 3 + a$ missing edges. By Corollary 2.5, if $d \geq 3$, there is Hamilton cycle in $G[N(u)]$ with $\alpha \leq a$ missing edges; i.e. there is an ordering $x_1, \dots, x_d$ of the vertices in $N(u)$, such that $x_i x_{i+1}$ is an edge in G for all but α values of $i \in [d]$ (addition is taken modulo d). Let ω' be the fractional triangle packing that gives each triangle $u x_i x_{i+1}$ with $x_i x_{i+1} \in E(G)$ weight $1/2$, and let G' be the weighted graph obtained from $G \setminus \{u\}$ by giving $x_i x_{i+1}$ weight $1/2$ whenever $x_i x_{i+1}$ is an edge of G; giving weight 1 to every other edge of $G \setminus \{u\}$; and giving non-edges weight 0. The total missing weight in G' is at most

$$\begin{aligned}
n-4+a-\bar{d}+(d-\alpha)/2 &= n-4+a-\alpha/2+(n-1-\bar{d})/2-\bar{d} \\
&= 3n/2-4.5+a-\alpha/2-3\bar{d}/2 \\
&\le (3n/2-4.5+a-\alpha/2)-(n+\alpha+1)/2 \\
&= n-5+a-\alpha \\
&= (n-1)-4+(a-\alpha),
\end{aligned}$$

using $3\bar{d} \ge n+a+1 \ge n+\alpha+1$ for the inequality. By the induction hypothesis together with Lemma 2.4, it follows that there is a fractional triangle packing in G' with no heavy triangles and with uncovered weight at most $a-\alpha$. Combining this packing with ω, we obtain a fractional triangle packing of G with uncovered weight at most a and no heavy triangles, as required.

It remains to consider the case where $d \in \{0,1,2\}$. If $d \in \{0,1\}$ then $\bar{d} \ge n-2$, so $a \ge 2$. The graph $G \setminus \{u\}$ has at most two missing edges, so by induction it has a fractional triangle decomposition ω', which is a triangle packing in G with uncovered weight at most $1 \le a$. If $d=2$ then $\bar{d} \ge n-3$, so $a \ge 1$. If $a \ge 2$, we can again apply the induction hypothesis to conclude that there is a fractional triangle decomposition in G', which is a fractional triangle packing in G with uncovered weight at most $2 \le a$. Finally if $d=2$ and $a=1$, then $G[N(u)]$ consists of two adjacent vertices. We think of the single edge in $G[N(u)]$ as a Hamilton cycle with one missing edge, and repeat the above argument.

From now on, we assume that $\bar{d}(u) \le (n+a)/3$ for every vertex u. Let Z be the set of vertices u with $\bar{d}(u)=0$, let $U := V(G) \setminus Z$, and write $m := |Z|$.

5.2 Case 2. $m \in \{0,1,2,3\}$

Let K be the set of vertices u with $\bar{d}(u) \ge 3$, let L be the set of vertices u with $\bar{d}(u)=2$, and denote $k := |K|$ and $\ell := |L|$.

Claim 5.1. $2k+\ell \ge m+3$.

Proof. Suppose that $2k+\ell \le m+2$. Then

$$\begin{aligned}
2(n-4+a) = \sum_{u\in U} \bar{d}(u) &\le k\cdot\frac{n+a}{3}+2\ell+n-m-k-\ell \\
&\le k\cdot\frac{n+a}{3}+m-2k+2+n-m-k \\
&= k\cdot\frac{n+a}{3}+2-3k+n.
\end{aligned}$$

It follows that

$$n \le \frac{30+(k-6)\cdot a-9k}{3-k}.$$

If $k = 0$, we obtain $n \le 10 - 2a \le 10$; if $k = 1$, we have $n \le 15 - 2.5a - 4.5 \le 10.5$; and if $k = 2$, we have $n \le 30 - 4a - 18 \le 12$. Either way, we reach a contradiction to the assumption that $n \ge 14$, thus proving the claim. $\square$

Let M_1, M_2 be two edge-disjoint matchings between Z and $K \cup L$ that cover Z, such that every vertex in L is covered by at most one of the two matchings. By Claim 5.1, such a matching exists. Indeed, by Claim 5.1 (using $m \le 3$), there exist sets $S_1, S_2 \subseteq K$ and $T_1, T_2 \subseteq L$, such that T_1 and T_2 are disjoint, $|S_i| + |T_i| = m$ for $i \in [2]$, and S_1 and S_2 are disjoint if $m = 1$. Now take M_1 to be any perfect matching in $G[Z, S_1 \cup T_1]$, and take M_2 to be any perfect matching in $G[Z, S_2 \cup T_2] \setminus M_1$. Write $d_1(u)$ and $d_2(u)$ for the degree of u in M_1 and M_2, respectively.

For $u \in U$ let G_u be the graph obtained from G by removing u, removing z if $uz \in M_1$ for some $z \in Z$, and removing the edge uz if $uz \in M_2$ for some $z \in Z$ (note that at most two vertices and at most one edge are removed). Define $r(u) = \min\{a, \bar{d}(u) - 1 - d_1(u) - d_2(u)\}$; so $r(u) \ge 0$ for every vertex u, by choice of M_1 and M_2. By definition of M_1 and M_2, the graph G_u has $n - 1 - d_1(u)$ vertices and the following number of missing edges

$$\begin{aligned} n - 4 + a - \bar{d}(u) + d_2(u) &= n - 5 - d_1(u) + \big(a - \big(\bar{d}(u) - 1 - d_1(u) - d_2(u)\big)\big) \\ &\le n - 5 - d_1(u) - (a - r(u)). \end{aligned}$$

(Here we used the assumption that M_1 and M_2 are edge-disjoint.) By induction, there is a fractional triangle packing ω_u in G_u with uncovered weight at most $a - r(u)$ that has no heavy triangles. Take $\omega = \frac{1}{|U|-2} \sum_{u \in U} \omega_u$. Note that every edge in G appears in exactly $|U| - 2$ of the graphs G_u. It follows that the uncovered weight of ω is at most

$$\frac{1}{|U| - 2} \sum_{u \in U} (a - r(u)) \le \frac{1}{|U| - 2} \cdot (|U|a - \sum_{u \in U} r(u)) \le a,$$

where for the second inequality we used the following claim. As every triangle appears in at most $|U| - 2$ of the graphs G_u, there are no heavy triangles in ω. The proof of Theorem 2.1 in this case would be completed once the following claim is proved.

Claim 5.2. *$\sum_{u \in U} r(u) \ge 2a$.*

Proof. Suppose that $\sum_{u \in U} r(u) \le 2a - 1$. As $r(u) \ge 0$ for every $u \in U$, we have $a \ge 1$.

Suppose first that $\bar{d}(u) \le a + 1 + d_1(u) + d_2(u)$ for every vertex $u \in U$. Then

$$\begin{aligned} 2a - 1 \ge \sum_{u \in U} r(u) &= \sum_{u \in U} (\bar{d}(u) - 1 - d_1(u) - d_2(u)) \\ &= 2(n - 4 + a) - (n - m) - 2m \\ &= n - 8 + 2a - m. \end{aligned}$$

It follows that $n \le 7 + m \le 10$, a contradiction to $n \ge 14$.

Now suppose that $\bar{d}(v) \geq a+2+d_1(v)+d_2(v)$ for some vertex v, implying that $\bar{d}(u) \leq a+1+d_1(u)+d_2(u)$ for every $u \in U \setminus \{v\}$ (as otherwise $\sum_u r(u) \geq 2a$). So

$$\begin{aligned} 2a-1 \geq \sum_{u \in U} r(u) = a + \sum_{u \in U \setminus \{v\}} (\bar{d}(u) - 1 - d_1(u) - d_2(u)) \\ \geq a + 2(n-4+a) - \frac{n+a}{3} - (n-m-1) - 2m \\ = \frac{2n}{3} + 2a + \frac{2a}{3} - 7 - m. \end{aligned}$$

It follows that $n \leq 9 - a + \frac{3m}{2} \leq 13.5$, a contradiction. $\square$

5.3 Case 3. $a < 4$, $m \geq 4$

For $z \in Z$, let ω_z be a fractional triangle packing in $G \setminus \{z\}$ with no heavy triangles and with uncovered weight (exactly) $a+1$ (such a packing exists by induction). We assume that ω_z is symmetric on Z, i.e. swapping the roles of any two vertices in Z does not affect ω_z (this can be achieved by averaging over all packings obtained by permutations of $Z \setminus \{z\}$). Similarly, we assume that $\omega_{z'}$ can be obtained from ω_z by swapping the roles of z and z', for every $z, z' \in Z$. Let ϕ_z be an edge-weighting, of total weight 1, corresponding to weight uncovered by ω_z (namely, $\sum_{e \in E(G \setminus \{z\})} \phi_z(e) = 1$, and for every edge e in $G \setminus \{z\}$, $\omega_z(e) + \phi_z(e) \leq 1$); we again assume that ϕ_z is symmetric with respect to Z. Let ψ_z be a weighting on $G \setminus \{z\}$ defined by $\psi_z(e) = 1 - \omega_z(e) - \phi_z(e)$ for every edge e in $G \setminus \{z\}$. Write $\gamma := \phi_z(zz')$ for some distinct $z, z' \in Z \setminus \{z\}$; $\alpha_u = \phi_z(uz)$ for $u \in U$ and $z \in Z \setminus \{z\}$; and $\beta_e = \phi_z(e)$ for every edge e in U (note that γ and α_u are well-defined, by the symmetry with respect to Z). Define $\alpha = (m-1)\sum_{u \in U} \alpha_u$ and $\beta = \sum_{e \in E(G[U])} \beta_e$. Then

$$\binom{m-1}{2}\gamma + \alpha + \beta = 1. \tag{1}$$

In order to find the required fractional triangle packing in G, we use two approaches. In the first one we consider the graphs G_u for $u \in U$, and modify them slightly by reducing the weight of some edges incident with vertices of Z, taking $\bar{d}(u)$ into account; in particular, the larger $\bar{d}(u)$ is, the more weight we can remove while still being able to use the induction hypothesis. We then use the available weight on edges incident with Z to compensate for the weight encoded by ϕ_z, to end up with a packing that has at most a uncovered weight (in contrast to the $a+1$ bound for ω_z). This approach works when m is not too large, because the larger m is, the more extra weight we need to compensate for.

In the second approach we use the edges in $U \times Z$ to compensate for the extra weight encoded by β_e for $e \in E(G[U])$, and then cover the remaining weight on these cross edges using triangles with at least two vertices in Z. This approach works for larger m, because as m grows, the ratio between the weight on edges in Z and the weight on edges in $U \times Z$ increases.

Define $r(u) = \min\{\bar{d}(u) - 1, a\}$.

Claim 5.3. $\sum_{u \in U} r(u) \geq 2a$.

Proof. Suppose that $\sum_u r(u) \leq 2a - 1$. Let k be the number of vertices $u \in U$ with $\bar{d}(u) \geq a + 1$; then $k \leq 1$. We have

$$\begin{aligned}
2a - 1 \geq \sum_{u \in U} r(u) \geq \sum_{u \in U} (\bar{d}(u) - 1) - k\left(\frac{n+a}{3} - 1\right) + ka \\
= 2(n - 4 + a) - (n - m) - \frac{kn}{3} - \frac{ka}{3} + k + ka \\
= \frac{(3-k)n}{3} - (8 - k) + 2a + \frac{2ka}{3} + m.
\end{aligned}$$

If $k = 0$, we obtain $n \leq 7 - m \leq 7$; and if $k = 1$, we obtain $n \leq 9 - 3m/2 - a \leq 9$. Either way, this is a contradiction to $n \geq 14$. $\square$

Let $\sigma : U \to \mathbb{Z}^{\geq 0}$ be such that $\sigma(u) \leq r(u)$ and $\sum_u \sigma(u) = 2a$; such a weight assignment exists by Claim 5.3. Let H be an auxiliary bipartite graph, with vertex sets X and Y, where $X = \{u_0 : u \in U\} \cup \{\zeta\}$, and $Y = \{u_1 : u \in U\}$, and edge set $X \times Y \setminus \{u_0 u_1 : u \in U\}$. We think of u_0 and u_1 as representing u, and of ζ as representing Z. We assign a weight $\tau(x)$ to every vertex $x \in V(H)$, as follows.

$$\tau(x) = \begin{cases} m \sum_{v \in U} \beta_{uv} & x = u_0 \text{ for some } u \in U \\ \frac{m}{2} \cdot \alpha & x = \zeta \\ \bar{d}(u) - 1 - \sigma(u) & x = u_1 \text{ for some } u \in U. \end{cases}$$

(If uv is not an edge, $\beta_{uv} = 0$.)

A *fractional matching* in H is an assignment $\nu : E(H) \to \mathbb{R}^{\geq 0}$ such that $\sum_{w \in V(H)} \nu(vw) \leq \tau(v)$ for every $v \in V(H)$. We say that a fractional matching ν *saturates* V if $\sum_{w \in V(H)} \nu(vw) = \tau(v)$ for every $v \in V$.

Claim 5.4. *If $m \leq n - 8$, then there is a fractional matching in H that saturates X.*

Proof. By a fractional version of Hall's theorem, it suffices to show that every set $A \subseteq X$ satisfies $\tau(N(A)) \geq \tau(A)$. As $N(A) = Y$ for every $A \subseteq X$ except for $A = \emptyset$ or $A = \{u_0\}$ for some $u \in U$, it suffices to check that $\tau(Y) \geq \tau(X)$ and $\tau(Y \setminus \{u_1\}) \geq \tau(u_0)$ for every $u \in U$.

$$\begin{aligned}
\tau(Y) &= \sum_{u \in U} (\bar{d}(u) - 1 - \sigma(u)) = 2(n - 4 + a) - (n - m) - 2a = n + m - 8. \\
\tau(X) &= m \sum_{u \in U} \sum_{v \in U} \beta_{uv} + \frac{m}{2} \cdot \alpha = 2m\beta + \frac{m}{2} \cdot \alpha \leq 2m,
\end{aligned}$$

by (1). Thus, as $m \leq n - 8$, we have $\tau(Y) \geq \tau(X)$.

Fix $u \in U$. Then

$$\tau(u_0) = m \sum_{v \in U} \beta_{uv} \le m,$$

$$\tau(Y \setminus \{u_1\}) \ge \tau(Y) - \bar{d}(u) + 1 \ge n + m - 8 - \frac{n+a}{3} + 1 \ge \frac{2n}{3} + m - 8 \ge m,$$

using (1), $2n/3 \ge 28/3 > 8$ and $a \le 3$. Thus, $\tau(Y \setminus \{u_1\} \ge \tau(u_0)$ for every $u \in U$, completing the proof of Claim 5.4. $\square$

Consider a fractional matching as in Claim 5.4, and let $\nu_u(v)$ be the weight of the edge v_0u_1 for $u, v \in U$, and let $\nu_u(\zeta)$ be the weight of the edge ζu_1. Then

$$\sum_{v \in U} \nu_u(v) + \nu_u(\zeta) \le \tau(u_1) = \bar{d}(u) - 1 - \sigma(u),$$

$$\sum_{u \in U} \nu_u(v) = \tau(u_0) = m \sum_{u \in U} \beta_{uv},$$

$$\sum_{u \in U} \nu_u(\zeta) = \tau(\zeta) = \frac{m}{2} \cdot \alpha.$$

Let G_u be the weighted graph obtained from $G \setminus \{u\}$ by decreasing the weight of vz (from 1) by $\nu_u(v)/m$ for $v \in U \setminus \{v\}$ and $z \in Z$, and decreasing the weight of zz' by $\nu_u(\zeta)/\binom{m}{2}$ for every distinct $z, z' \in Z$. The missing weight in G_u is

$$\begin{aligned} n - 4 + a - \bar{d}(u) + \sum_{v \in U} \nu_u(v) + \nu_u(\zeta) &\le n - 4 + a - \bar{d}(u) + \bar{d}(u) - 1 - \sigma(u) \\ &= (n-1) - 4 + a - \sigma(u). \end{aligned}$$

Thus, by the induction hypothesis and Lemma 2.4, there is a fractional triangle packing ω_u in G_u with no heavy triangles and with uncovered weight at most $a - \sigma(u)$; let $\psi_u(e)$ be the uncovered weight at e, for any edge e in G_u.

Let ω' be a fractional triangle packing defined as follows, for distinct $u, v \in U$ and $z, z', z'' \in Z$,

$$\omega'(uvz) = \beta_{uv} \qquad \omega'(uzz') = \alpha_u \qquad \omega'(zz'z'') = \gamma,$$

and

$$\omega = \frac{1}{n-2}\left(\sum_{v \in V(G)} \omega_v + \omega'\right) \qquad \psi = \frac{1}{n-2} \sum_{v \in V(G)} \psi_v.$$

Claim 5.5.

(a) $\omega(e) + \psi(e) = 1$ *for every edge* e *in* G,

(b) $\sum_{e \in E(G)} \psi(e) \le a$.

Proof. Recall that $\sum_e \psi_z(e) \le a$ for every $z \in Z$, and $\sum_e \psi_u(e) \le a - \sigma(u)$ for $u \in U$. It follows that

$$\sum_{v \in V(G),\, e \in E(G)} \psi_v(e) \le na - \sum_u \sigma(u) = (n-2)a,$$

implying that $\sum_e \psi(e) \le a$, as required for (b).

Let e be an edge in $G[U]$. We consider three cases: $e = uv$ for $u, v \in U$; $e = uz$ for $u \in U$ and $z \in Z$; and $e = zz'$ for $z, z' \in Z$. In the first case,

$$\begin{aligned}
(n-2)(\omega(e) + \psi(e)) &= \sum_{z \in Z} (\omega_z(e) + \psi_z(e)) + \sum_{w \in U \setminus \{u,v\}} (\omega_w(e) + \psi_w(e)) + \omega'(e) \\
&= m(1 - \beta_e) + (n - m - 2) + m\beta_e \\
&= n - 2.
\end{aligned}$$

In the second case,

$$\begin{aligned}
(n-2)(\omega(e) + \psi(e)) = &\sum_{z' \in Z \setminus \{z\}} (\omega_{z'}(e) + \psi_{z'}(e)) + \sum_{v \in U \setminus \{u\}} (\omega_v(e) + \psi_v(e)) + \omega'(e) \\
&= (m-1)(1 - \alpha_u) + (n - m - 1) - \frac{1}{m} \sum_{v \in U \setminus \{u\}} \nu_v(u) + \sum_{v \in U \setminus \{u\}} \beta_{uv} + (m-1)\alpha_u \\
&= n - 2.
\end{aligned}$$

And in the third case,

$$\begin{aligned}
(n-2)(\omega(e) + \psi(e)) &= \sum_{z'' \in Z \setminus \{z, z'\}} (\omega_{z''}(e) + \psi_{z''}(e)) + \sum_{u \in U} (\omega_u(e) + \psi_u(e)) + \omega'(e) \\
&= (m-2)(1 - \gamma) + n - m - \frac{1}{\binom{m}{2}} \sum_u \nu_u(\zeta) + (m-2)\gamma + \sum_{u \in U} \alpha_u \\
&= n - 2.
\end{aligned}$$

We conclude that $\omega(e) + \psi(e) = 1$ for every $e \in E(G)$, as required for (a). □

By Claim 5.5, ω is a fractional triangle packing in G with uncovered weight at most a. There are no heavy triangles in ω, as every triangle appears in at most $n - 2$ of the packings ω' and ω_v for $v \in V(G)$, and none of these packings have a heavy triangle. This completes the proof of Theorem 2.1 in this case when $m \le n - 8$.

We now assume that $m \ge n - 7$.

Define, for distinct $u, v \in u$ and $z, z', z'' \in Z$,

$$\begin{aligned}
\omega'(uvz) &= \beta_{uv} \\
\omega'(uzz') &= \frac{1}{m-1}\left(1 + (m-1)\alpha_u - \sum_{w \in U} \beta_{uw}\right) \\
\omega'(zz'z'') &= \frac{1}{m-2}\left(2 + (m-2)\gamma - \frac{n-m}{m-1} - \frac{\alpha}{m-1} + \frac{2\beta}{m-1}\right).
\end{aligned} \tag{2}$$

Note that $\omega'(T) \geq 0$ for every triangle T in G. Indeed, this clearly holds for $T = uvz$ for some $u, v \in U$ and $z \in Z$, as $\beta_{uv} \geq 0$. Next, if $T = uzz'$ for $u \in U$ and $z, z' \in Z$, then, as $\sum_v \beta_{uv} \leq 1$ (by (1)), we indeed have $\omega'(T) \geq 0$. Finally, if $T = zz'z''$ for $z, z', z'' \in Z$, it suffices to show that

$$2 \geq \frac{n-m}{m-1} + \frac{\alpha}{m-1}.$$

As $\alpha \leq 1$ (by (1)), it suffices to show that

$$0 \leq 2(m-1) - (n-m) - 1 = 3m - 3 - n.$$

Recall that $m \geq n - 7$, so we have $3m - 3 - n \geq 2n - 24 > 0$, as required.

Define

$$\begin{aligned}
\omega &= \frac{1}{m}\left(\sum_{z \in Z} \omega_z + \omega'\right) \\
\psi &= \frac{1}{m}\sum_{z \in Z} \psi_z.
\end{aligned} \tag{3}$$

Claim 5.6.

(a) $\sum_{e \in E(G)} \psi(e) \leq a$,

(b) $\omega(e) + \psi(e) = 1$ *for every* $e \in E(G)$.

Proof. Recall that $\sum_{e \in E(G \setminus \{z\})} \psi_z(e) \leq a$ for every $z \in Z$, (a) follows from the definition of ψ.

For (b), we consider three cases: $e = uv$ with $u, v \in U$; $e = uz$ with $u \in U$ and $z \in Z$; and $e = zz'$ with $z, z' \in Z$. In each of these cases we will show that $m \cdot \omega(e) = m$. In the first case we have

$$m \cdot \omega(e) = m(1 - \beta_e) + m\beta_e = m.$$

In the second case,

$$m \cdot \omega(e) = (m-1)(1 - \alpha_u) + \sum_{v \in U} \beta_{uv} + \left(1 + (m-1)\alpha_u - \sum_{v \in U} \beta_{uv}\right) = m.$$

And in the third case,

$$\begin{aligned} m \cdot \omega(e) = (m-2)(1-\gamma) + \frac{n-m}{m-1} + \alpha - \frac{2\beta}{m-1} \\ + 2 + (m-2)\gamma - \frac{n-m}{m-1} - \alpha + \frac{2\beta}{m-1} = m, \end{aligned}$$

completing the proof of Claim 5.6. □

5.4 Case 4. $a = 4$, $m \geq 4$

Fix a non-edge xy (so $x, y \in U$). For $z \in Z$, define G_z to be the graph obtained from G by removing the vertex z and adding the edge xy. So G_z has $n-1$ vertices and $n-5+a$ missing edges, thus by induction there is a fractional triangle packing ω'_z on G_z that has uncovered weight at most a, and has no heavy triangles. We assume that ω'_z is symmetric on $Z \setminus \{z\}$, and that ω'_z can be obtained from $\omega'_{z'}$ by swapping the roles of z and z' for every $z, z' \in S$. Let ψ_z be the edge-weighting corresponding to the weight uncovered by ω'_z. Let ϕ_z be the edge-weighting defined by $\phi_z(vx) = \phi_z(vy) = \omega'_z(vxy)$ for $v \in V(G) \setminus \{x, y\}$. Let ω_z be the fractional triangle packing obtained from ω'_z by changing the weight of triangles containing xy to 0. Define $\gamma = \phi_z(z'z'')$, $\alpha_u = \phi_z(uz')$, and $\beta_{uv} = \phi_z(uv)$, for distinct $z, z', z'' \in Z$ and distinct $u, v \in U$, and write $\alpha = (m-1)\sum_{u \in U} \alpha_u$ and $\beta = \sum_{e \in E(G[U])} \beta_e$. Then

(i) $\omega_z(e) + \psi_z(e) + \phi_z(e) = 1$ for every $e \in E(G \setminus \{z\})$.

(ii) $\sum_{e \in E(G[U])} \psi_z(e) \leq a$.

(iii) $\sum_{e \in E(G[U])} \phi_z(e) = \binom{m-1}{2}\gamma + \beta + \alpha \leq 2$.

(iv) $\sum_{v \in U} \beta_{uv} \leq 1$ for every $u \in U$.

To see (iii), note that $\sum_v \omega'_z(vxy) \leq 1$, thus $\sum_e \phi_z(e) = 2\sum_v \omega'_z(vxy) \leq 2$. Let $u \in U$. If $u = x$ or $u = y$, then $\sum_v \beta_{uv} = \sum_v \omega'_z(vxy) \leq 1$; and if $u \neq x, y$, then $\sum_v \beta_{uv} = 2\omega'_z(uxy) \leq 1$, by the assumption that ω'_z does not have heavy triangles; (iv) follows. We note that (iv) is the reason why we introduced the assumption that there are no heavy triangles.

We follow the two approaches introduced in the previous case. One main difference is the definition of ϕ_z (which is necessary because we cannot use the induction hypothesis for $a+1$, as we did in the previous case), which manifests itself in the upper bound of 2 in (iii), replacing the upper bound of 1 that we had previously. This implies that in the first approach we need to compensate for more 'extra' weight, thus restricting the range of m's for which the approach works. In order to cover all possible values of m, we capitalise on the larger value of a, which allows us to remove more weight from the graphs $G \setminus \{u\}$ with $u \in U$. The exact details make this case somewhat technical. For convenience, we reverse the order of the two approaches.

Let ω' be a fractional triangle packing defined as in (2) from the previous case. We note that $\omega'(T) \geq 0$ for every triangle T with at least one vertex in U. Indeed, as $\beta_{uv} \geq 0$ for every $u, v \in U$, this holds for T with two vertices in U; and if T has one vertex in U, the non-negativity follows from (iv). If T has three vertices in Z, then $\omega'(T) \geq 0$ if $2(m-1) - (n-m) - \alpha + 2\beta \geq 0$. As $\alpha + \beta \leq 2$, it suffices to have

$$3m \geq n + 4 - 3\beta. \tag{4}$$

If (4) holds, we define ω and ψ as in (3). The proof of Theorem 2.1 can then be completed following the proof of Claim 5.6. Thus, from now on, we assume that (4) does not hold.

As before, put $r(u) = \min\{a, \bar{d}(u) - 1\}$.

Claim 5.7. $\sum_{u \in U} r(u) \geq 2a$. *Moreover, if* $3m \geq n - 7$ *then* $\sum_{u \in U} r(u) \geq 2a + 6$.

Proof. The proof of Claim 5.3 can be repeated here to show that $\sum_{u \in U} r(u) \geq 2a$.

For the second part, suppose that $\sum_{u \in U} r(u) \leq 2a + 5$. Let k be the number of vertices u with $\bar{d}(u) \geq a + 1$. If $k \geq 4$ we have $\sum_{u \in U} r(u) \geq 4a \geq 2a + 5$ (as $a = 4$), so we assume that $k \leq 3$.

$$\begin{aligned}
2a + 5 \geq \sum_{u \in U} r(u) &\geq ka + \sum_{u \in U} (\bar{d}(u) - 1) - k\left(\frac{n+a}{3} - 1\right) \\
&= \frac{2ka}{3} + 2(n - 4 + a) - (n - m) - \frac{kn}{3} + k \\
&\geq \frac{2ka}{3} + \frac{(3-k)n}{3} - 8 + k + 2a + \frac{n-7}{3} \\
&= \frac{2ka + (4-k)n - 31 + 3k}{3} + 2a,
\end{aligned}$$

using $3m \geq n - 7$. It follows that

$$(4 - k)n \leq 46 - 2ka - 3k.$$

If $k = 0$ we obtain $4n \leq 46$; if $k = 1$, we have $3n \leq 46 - 2a - 3 = 35$; if $k = 2$, we have $2n \leq 46 - 4a - 6 = 24$; and if $k = 3$, we obtain $n \leq 46 - 6a - 9 = 13$. Either way, as $n \geq 14$, we reached a contradiction. $\square$

Define

$$\rho = \begin{cases} 0 & 3m \leq n - 8 \\ \min\{6, m\beta\} & 3m \geq n - 7. \end{cases}$$

Let σ be a function $\sigma : U \to \mathbb{Z}^{\geq 0}$ such that $\sigma(u) \leq r(u)$ for every $u \in U$ and $\sum_{u \in U} \sigma(u) = 2a + \lceil \rho \rceil$; note that such σ exists by Claim 5.7.

Let H be a bipartite auxiliary graph with vertex sets $X := \{u_0 : u \in U\} \cup \{\zeta\}$ and $Y := \{u_1 : u \in U\}$

and edges $(X \times Y) \setminus \{u_0u_1 : u \in U\}$. Define

$$\tau(x) = \begin{cases} m\left(1 - \frac{\rho}{\beta m}\right) \cdot \sum_{v \in U} \beta_{uv} & x = u_0 \text{ for some } u \in U \\ \frac{m}{2} \cdot \alpha & x = \zeta \\ \bar{d}(u) - 1 - \sigma(u) & x = u_1 \text{ for some } u \in U. \end{cases}$$

Claim 5.8. *There is a fractional matching in H that saturates X.*

Proof. As in the proof of Claim 5.4, in order to prove that the required matching exists, it suffices to show that $\tau(Y) \geq \tau(X)$ and $\tau(Y \setminus \{u_1\}) \geq \tau(u_0)$ for every $u \in U$.

$$\tau(Y) = \sum_{u \in U} (\bar{d}(u) - 1 - \sigma(u)) = 2(n - 4 + a) - (n - m) - 2a - \lceil \rho \rceil = n + m - 8 - \lceil \rho \rceil$$

$$\tau(X) = m\left(1 - \frac{\rho}{\beta m}\right) 2\beta + \frac{m}{2} \cdot \alpha = 2m\beta - 2\rho + \frac{m}{2} \cdot \alpha \leq 4m - 2\rho.$$

If $3m \leq n - 8$ and $\rho = 0$, we have

$$\tau(Y) - \tau(X) \geq n + m - 8 - 4m \geq 0,$$

as required. If $3m \geq n - 7$ and $\rho = 6$, we have

$$\begin{aligned} \tau(Y) - \tau(X) &= n + m - 14 - 2m\beta + 12 - \frac{m}{2} \cdot \alpha \\ &\geq n + m - 2 - 2m\beta - \frac{m}{2} \cdot (2 - \beta) \\ &\geq \frac{(2(n-2) - 3m\beta)}{2} \end{aligned}$$

By the assumption that (4) does not hold, we have

$$3m\beta \leq \beta(n + 4 - 3\beta) \leq 2(n - 2),$$

where the last inequality holds as $\beta(n + 4 - 3\beta)$ is increasing when $\beta \in [0, 2]$ (the derivative is $n + 4 - 6\beta \geq n - 8 > 0$), and is thus maximised at $\beta = 2$. It follows that $\tau(Y) \geq \tau(X)$ in this case. Finally, if $\rho = \beta m$, we have $\tau(u_0) = 0$ for every $u \in U$. Hence,

$$\tau(Y) - \tau(X) = n + m - 8 - \lceil \rho \rceil - \frac{\alpha m}{2} \geq n + m - 14 - m \geq 0,$$

where we used the inequalities $\rho \leq 6$, $n \geq 14$ and $\alpha \leq 2$. We have thus verified that $\tau(Y) \geq \tau(X)$ for all possible values of ρ.

We now show that $\tau(Y \setminus \{u_1\}) \geq \tau(u_0)$ for every $u \in U$. Note that when $\rho = \beta m$, $\tau(u_0) = 0$ for every $u \in U$, so this folds trivially. Next, suppose that $\rho \in \{0, 6\}$. Then, using (iv),

$$\tau(u_0) = \left(m - \frac{\rho}{\beta}\right) \sum_{w \in U} \beta_{uw} \leq m - \frac{\rho}{2}.$$

Thus, if $\rho = 0$,

$$\begin{aligned} \tau(Y \setminus \{u_1\}) - \tau(u_0) &= \tau(Y) - \bar{d}(u) + 1 + \sigma(u) - m \\ &\geq n + m - 8 - \frac{n+a}{3} + 1 - m \\ &= \frac{2n}{3} - \frac{a}{3} - 7 \\ &\geq \frac{2n - 25}{3} \geq 0, \end{aligned}$$

as $n \geq 14$. Finally, consider the case $\rho = 6$. Before continuing, we modify σ, and before that, we note that there are at most eight vertices u with $\bar{d}(u) \geq (n+a)/3 - 2$. Indeed, otherwise

$$2n = 2(n - 4 + a) = \sum_{u \in U} \bar{d}(u) \geq 8\left(\frac{n+a}{3} - 2\right) = \frac{8n - 16}{3},$$

a contradiction to $n \geq 14$. We now modify σ so that $\sigma(u) \geq 2$ for every $u \in U$ with $\bar{d}(u) \geq (n+a)/3 - 2$; and $\sum_u \sigma(u) = 2a + 6 = 14$ (by the above argument such σ exists). We thus have $\bar{d}(u) - \sigma(u) \leq (n+a)/3 - 2$ for every $u \in U$, so

$$\begin{aligned} \tau(Y \setminus \{u_1\}) - \tau(u_0) &= \tau(Y) - \bar{d}(u) + \sigma(u) + 1 - m + \frac{\rho}{2} \\ &\geq n + m - 8 - \rho - \frac{n+a}{3} + 3 - m + \frac{\rho}{2} \\ &= \frac{2n}{3} - \frac{a}{3} - 5 - \frac{\rho}{2} \\ &= \frac{2n - 28}{3} \geq 0, \end{aligned}$$

as $n \geq 14$, $a = 4$ and $\rho = 6$. $\square$

Consider a fractional matching as in Claim 5.8, define $\nu_u(v)$ to be the weight of the edge u_0v_1 for distinct $u, v \in U$, and define $\nu_u(\zeta)$ to be the weight of ζu_1 for $u \in U$. Then

$$\begin{aligned} &\sum_{v \in U} \nu_u(v) + \nu_u(\zeta) \leq \tau(u_1) = \bar{d}(u) - 1 - \sigma(u), \\ &\sum_{u \in U} \nu_u(v) = \tau(v_0) = \left(m - \frac{\rho}{\beta}\right) \sum_{u \in U} \beta_{uv}, \\ &\sum_{u \in U} \nu_u(\zeta) = \tau(\zeta) = \frac{m}{2} \cdot \alpha. \end{aligned}$$

Let G_u be the graph obtained from G by removing the vertex u; decreasing the weight of vz, where

$v \in U$ and $z \in Z$, by $\nu_u(v)/m$; and decreasing the weight of zz', where $z, z' \in Z$, by $\nu_u(\zeta)/\binom{m}{2}$. Note that the weights of the edges of G_u are non-negative, by (iv) and (iii). The missing weight in G_u is

$$\begin{aligned} n - 4 + a - \bar{d}(u) + \sum_{v \in U} \nu_u(v) + \nu_u(\zeta) &\le n - 4 + a - \bar{d}(u) + \bar{d}(u) - 1 - \sigma(u) \\ &= (n-1) - 4 + (a - \sigma(u)). \end{aligned}$$

Thus, by induction and by Lemma 2.4, there is a fractional triangle packing ω_u in G_u that has no heavy triangles and has uncovered weight at most $a - \sigma(u)$; let ψ_u be the weighting corresponding the to weight uncovered by ω_u.

Let ω' be the fractional triangle packing defined as follows, for distinct $u, v \in U$ and $z, z', z'' \in Z$.

$$\omega'(uvz) = \left(1 - \frac{\rho}{\beta m}\right)\beta_{uv} \qquad \omega'(uzz') = \alpha_u \qquad \omega'(zz'z'') = \gamma.$$

Let ψ' be the edge-weighting defined by $\psi'(e) = \rho\beta_e/\beta$ if $e = uv$ for $u, v \in U$, and setting $\psi(e) = 0$ otherwise. Define

$$\omega = \frac{1}{n-2}\left(\sum_{v \in V(G)} \omega_v + \omega'\right) \qquad \psi = \frac{1}{n-2}\left(\sum_{v \in V(G)} \psi_v + \psi'\right).$$

Claim 5.9.

(a) $\omega(e) + \psi(e) = 1$ *for every edge* e *in* G,

(b) $\sum_{e \in E(G)} \psi(e) \le a$.

Proof. Recall that $\sum_e \psi_v(e) \le a - \sigma(v)$ for every $v \in U$ (setting $\sigma(z) = 0$ for $z \in Z$). Thus

$$\begin{aligned} \sum_{v \in V(G),\, e \in E(G)} \psi_v(e) + \sum_{e \in E(G)} \psi'(e) &\le na - \sum_{u \in U} \sigma(u) + \sum_{e \in E(G[U])} \frac{\rho\beta_e}{\beta} \\ &= na - 2a - \lceil \rho \rceil + \rho \le (n-2)a, \end{aligned}$$

thus proving (b). The rest of the proof is very similar to that of Claim 5.5; we omit further details. □

This completes the proof of Theorem 2.1. □

References

[1] B. Barber, D. Kühn, A. Lo, and D. Osthus, *Edge-decompositions of graphs with high minimum degree*, Adv. Math. **288** (2016), 337–385.

[2] M. Delcourt and L. Postle, *Progress towards Nash-Williams' Conjecture on Triangle Decompositions*, arXiv:1909.00514 (2019).

[3] F. Dross, *Fractional triangle decompositions in graphs with large minimum degree*, SIAM J. Discr. Math. **30** (2015), 36–42.

[4] P. Dukes, *Rational decomposition of dense hypergraphs and some related eigenvalue estimates*, Linear Algebra and its applications **436** (2012), 3726–3746, (see arXiv:1108.1576 for an erratum).

[5] P. Erdős, *Some recent problems and results in graph theory*, Discr. Math. **164** (1997), 81–85.

[6] P. Erdős and T. Gallai, *Graphs with given degree of vertices*, Mat. Lapok [Hungarian] **11** (1960), 264–274.

[7] K. Garaschuk, *Linear methods for rational triangle decompositions*, Ph.D. thesis, University of Victoria, 2014.

[8] V. Gruslys and S. Letzter, *Monochromatic triangle packings in red-blue graphs*, arXiv:2008.05311.

[9] T. Gustavsson, *Decompositions of large graphs and digraphs with high minimum degree*, Ph.D. thesis, University of Stockholm, 1991.

[10] P. Haxell and V. Rödl, *Integer and fractional packings in dense graphs*, Combinatorica **21** (2001), 13–38.

[11] B. D. McKay and A. Piperno, *Practical Graph Isomorphism, II*, J. Symbolic Computation **60** (2013), 94–112.

[12] C. S. J. Nash-Williams, *An unsolved problem concerning decomposition of graphs into triangles*, Combinatorial Theory and its Applications **III** (1070), 1179–1183.

[13] O. Ore, *Note on Hamilton Circuits*, Amer. Math. Monthly **67** (1960), 55–55.

[14] R. Yuster, *Asymptotically optimal K_k-packings of dense graphs via fractional K_k-decompositions*, J. Combin. Theory Ser. B **95** (2005), 1–11.