

Pancyclicity of highly connected graphs

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Abstract

A well-known result due to Chvatál and Erdős (1972) asserts that, if a graph G satisfies $\kappa(G) \geq \alpha(G)$, where $\kappa(G)$ is the vertex-connectivity of G , then G has a Hamilton cycle. We prove a similar result implying that a graph G is *pancyclic*, namely it contains cycles of all lengths between 3 and $|G|$: if $|G|$ is large and $\kappa(G) > \alpha(G)$, then G is pancyclic. This confirms a conjecture of Jackson and Ordaz (1990) for large graphs, and improves upon a very recent result of Draganić, Munhá-Correia, and Sudakov.

1 Introduction

A graph G is *Hamiltonian* if it contains a Hamilton cycle, namely a cycle through all the vertices of G . The notion of Hamiltonicity is a key notion in combinatorics and computer science (see the surveys [18, 23]). The decision problem, of determining whether a given graph is Hamiltonian, is one of 21 problems that have been shown to be NP-complete in Karp's influential paper [20] from 1972. As such, it is interesting to find simple sufficient conditions for Hamiltonicity. Perhaps the earliest and best-known example is Dirac's theorem [11] from 1952, which asserts that every n -vertex graph with minimum degree at least $n/2$ (and $n \geq 3$) has a Hamilton cycle.

We say that a graph G is *pancyclic* if it contains a cycle of length ℓ , for every $\ell \in [3, |G|]$. In 1971 Bondy [4] showed that Dirac's theorem can be strengthened substantially, to show that every graph on n vertices with minimum degree at least $n/2$ is either pancyclic, or a balanced complete bipartite graph. While interesting in its own right, Bondy believed that this example was a special case of a much more general phenomenon: in 1973 he [5] posed a famous 'meta-conjecture', asserting that almost every non-trivial property implying Hamiltonicity also implies pancyclicity, up to a small number of exceptions.

Since then, the validity of this meta-conjecture has been proved in many cases. For example, Bauer and Schmeichel [2] considered three properties involving degrees of vertices that were shown to imply Hamiltonicity by Bondy [6], Chvatál [8], and Fan [17], and showed that they also imply pancyclicity, with the exception of some bipartite graphs. Clark [10] (and later, independently, Shi [28] and Zhang [30]) showed that connected, locally connected claw-free graphs on at least three vertices are pancyclic,

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generalising a result of Oberly and Sumner [26], who proved that such graphs are Hamiltonian. Another very recent illustration of the validity of Bondy’s meta-conjecture is a result of Dragaić, Munhá-Correia, and Sudakov [13]. To state their result, let $\tilde{\alpha}(G)$ be the *bipartite independence number* of G , defined to be the maximum of $a + b$, over all a, b such that there exist disjoint sets A and B , of a and b vertices, such that G has no A - B edges. (In particular, $\tilde{\alpha}(G) \geq \alpha(G)$ for every graph G .) They proved that if a graph G satisfies $\delta(G) \geq \tilde{\alpha}(G)$, then G is pancyclic, unless G is a balanced complete bipartite graph, generalising a result of McDiarmid and YOLOV [25] who proved that this condition implies Hamiltonicity.

The result of McDiarmid and YOLOV is reminiscent of the following classic result of Chvátal and Erdős [9] from 1972: every graph G satisfying $\kappa(G) \geq \alpha(G)$, where $\kappa(G)$ is the vertex-connectivity of G , is Hamiltonian. In light of the above discussion, it is natural to ask: is (almost) every graph G satisfying $\kappa(G) \geq \alpha(G)$ pancyclic? It is easy to come up with a counterexample; indeed, any balanced complete bipartite graph is one. In fact, Bauer, van den Heuvel, and Schmeichel [3] found many examples of non-bipartite, triangle-free graphs G satisfying $\kappa(G) = \alpha(G)$. Such graphs are counterexamples to the question, due to the triangle-freeness. Nevertheless, Lou [24] showed that every such graph G contains a cycle of length ℓ for every $\ell \in [4, |G|]$, so the lack of triangles is the only barrier to pancyclicity for this class of examples. It is plausible that every graph G which satisfies $\kappa(G) \geq \alpha(G)$ and contains a triangle is pancyclic. In this direction, in 1986 Jackson and Ordaz [19] conjectured that every graph G with $\kappa(G) > \alpha(G)$ is pancyclic.

A result of Erdős [15] from 1974 implies that the stronger condition $\kappa(G) \geq 4(\alpha(G) + 1)^4$ guarantees pancyclicity. Amar, Fournier, and Germa [1] proved Jackson and Ordaz’s conjecture when $\alpha(G) \in \{2, 3\}$. Keevash and Sudakov [22] proved the conjecture up to a constant factor: they showed that if $\kappa(G) \geq 600\alpha(G)$ then G is pancyclic. Very recently, Draganić, Munhá-Correia, and Sudakov [14] proved it asymptotically: they showed that for every $\varepsilon > 0$, if $|G|$ is large enough and $\kappa(G) \geq (1 + \varepsilon)\alpha(G)$, then G is pancyclic. In this paper we prove Jackson and Ordaz’s conjecture for sufficiently large graphs.

Theorem 1.1. *There exists n_0 such that every graph G on at least n_0 vertices, that satisfies $\kappa(G) > \alpha(G)$, is pancyclic.*

We use many ingredients that are also used in [12] in the proof of the asymptotic version of the conjecture, including lemmas from [12, 14] about extending and shortening paths by a small amount, the main result in [12], upper bounds on cycle-complete Ramsey numbers [16, 21], and an upper bound on the Turán numbers of even cycles [7]. Nevertheless, in each step of the proof, new ingredients were needed. Here we highlight one such ingredient, which is a so-called rotation-extension argument, and might have applications elsewhere.

A new rotation argument. A very useful tool in the study of cycles in graphs is the so-called *rotation-extension* technique, introduced by Pósa [27] in the study of Hamiltonicity of random graphs, and later and independently also used by Thomason [29]. A standard rotation-extension argument proceeds as follows. Consider a cycle C , and for a vertex u in C , denote by u^- the predecessor of u in C (according to an arbitrary orientation of C). If u, v are two vertices in C , which can be joined by a path P with non-empty interior that lies outside of C , and u^-v^- is an edge, then we can form a new cycle $(uC_{u \rightarrow v}v^-u^-C_{u^- \leftarrow v}vPxPyu)$ (see Figure 2) which extends C . Suppose now that our underlying

graph has independence number α . Then, using this argument, if U is a set of at least $\alpha + 1$ vertices in C , any two of whose vertices can be joined by a path as above, then we can find an appropriate extension of C by the existence of an edge u^-v^- .¹ This approach fails, however, even when we can only find such a U of size slightly below α , as there is no way to guarantee the existence of an edge u^-v^- . Our new idea here is to consider the set $\{u^-, u^{-3} : u \in U\}$, where $u^{-3} = ((u^-)^-)^-$. If $|U| > \alpha/2$, and no two vertices of U are too close on C , then this is a set of size larger than α , which thus spans an edge. If that edge is of form u^-v^- , u^-v^{-3} , or $u^{-3}v^{-3}$, with distinct $u, v \in U$, and u, v can be joined by a path of length at least 6 with interior outside of C , then we can extend C similarly to the above (see Figures 3, 5a and 5b). The only other possibility is an edge of form u^-u^{-3} , which is a chord of length 2. Under the right conditions, this either gives us the desired extension of C , or yields many non-intersection chords of length 2, which can be utilised in other ways.

Organisation of the paper. In the next section, Section 2, we state three lemmas, corresponding to three ranges of cycles lengths (upper, middle, and lower), and show how to prove our main result, Theorem 1.1, using these lemmas (and an additional result due to Draganić, Munhá-Correia, and Sudakov [12]). In Section 3 we mention notation and preliminary results that will be used in the proofs of more than one of the three lemmas. We then prove the three lemmas in Sections 4 to 6, devoting one section to each lemma. We conclude the paper in Section 7 with some remarks regarding potential future research.

2 Proof of the main theorem

Our proof of Theorem 1.1 splits into three lemmas, according to the length of the cycles they can guarantee. Here are these three lemmas. The notation $a \ll b$ means that a is chosen to be sufficiently small with respect to b .

Lemma 2.1 (Upper range). *Let $0 < \delta \ll 1$ and let $n, \alpha, \ell \geq 1$ be integers, satisfying: α is sufficiently large, $\alpha \leq n \leq 4\alpha^2$, and $n/\delta\alpha \leq \ell \leq n$. Suppose that G is a graph on n vertices with $\alpha(G) = \alpha$ and $\kappa(G) > \alpha$. Then G has a cycle of length ℓ .*

Lemma 2.2 (Middle range). *Let $0 < \delta \ll 1$ and let $n, \alpha, \ell \geq 1$ be integers, satisfying: α is sufficiently large, $\sqrt{n}/2 \leq \alpha \leq \delta n^{2/3}$, and $n/\alpha \leq \ell \leq \delta(n/\alpha)^2$. Suppose that G is an n -vertex graph with $\alpha(G) = \alpha$ and $\kappa(G) \geq \alpha$. Then G contains a cycle of length ℓ .*

Lemma 2.3 (Lower range). *Let $0 < \delta \ll 1$ and let $n, \alpha, \ell \geq 1$ be integers, satisfying: α is sufficiently large, and $3 \leq \ell \leq \max\{n/\alpha, \delta\alpha\}$. Suppose that G is an n -vertex graph with $\delta(G) > \alpha(G) = \alpha$. Then G has a cycle of length ℓ .*

In addition, we shall need the following result, due to Draganić, Munhá-Correia, and Sudakov [12], allowing us to assume that the independence number is relatively large in terms of the number of vertices.

¹This approach proves the Chvatál–Erdős Hamiltonicity result mentioned above.

Theorem 2.4 (Draganić–Munha–Correia–Sudakov [12]). *Let $\varepsilon > 0$ and let n be sufficiently large. Suppose that G is a Hamiltonian graph on n vertices, satisfying $n \geq (2+\varepsilon)\alpha(G)^2$. Then G is pancyclic.*

The proof of our main result, Theorem 1.1, follows directly from the last four results.

Proof of Theorem 1.1. Let δ be a constant satisfying $0 < \delta \ll 1$. Let G be a graph on n vertices, where n is sufficiently large. Write $\alpha(G) = \alpha$, and let ℓ be an integer satisfying $3 \leq \ell \leq n$. We need to show that G contains a cycle of length ℓ .

First note that G is Hamiltonian, by the assumption $\kappa(G) > \alpha(G)$, and the well-known result of Chvátal and Erdős [9] mentioned in the introduction. By Theorem 2.4, if $n \geq 4\alpha^2$ then G is pancyclic, so in particular it contains a cycle of length ℓ .

So suppose that $n \leq 4\alpha^2$, and notice that at least one of the following holds: $\ell \geq n/\delta\alpha$; $n/\alpha \leq \ell \leq \delta(n/\alpha)^2$ and $\sqrt{n}/2 \leq \alpha \leq \delta n^{2/3}$; or $3 \leq \ell \leq \max\{n/\alpha, \delta\alpha\}$. Indeed, suppose that the first and third properties do not hold, namely $\max\{n/\alpha, \delta\alpha\} \leq \ell \leq n/\delta\alpha$. Then $\delta\alpha \leq n/\delta\alpha$, implying that $\alpha \leq \sqrt{n}/\delta \leq \delta n^{2/3}$. Thus also $n/\alpha \leq \ell \leq n/\delta\alpha \leq \delta(n/\alpha)^2$. This shows that the second property holds. Hence, we can apply one of Lemmas 2.1 to 2.3 to show that G contains a cycle of length ℓ . \square

3 Preliminaries

We will prove each of Lemmas 2.1 to 2.3 in a separate section, and mention the relevant preliminaries in these sections. Here we first define some notation, and then mention a few preliminaries that will be used in more than one of these sections.

Chords. A *chord* in a path P is an edge joining two non-consecutive vertices in P . The *length* of a chord is the length of the subpath of P between the two vertices of the chord. We call two chords *non-intersecting* if the interiors of the subpaths of P between the two vertices of each chord are disjoint. We will also use these notions with respect to a cycle C (when talking about the subpath of C between the vertices of the chord, we mean the shorter of these two subpaths, and in fact we will only consider chords of length 2 or 3). See Figure 1 for an illustration of this notion.

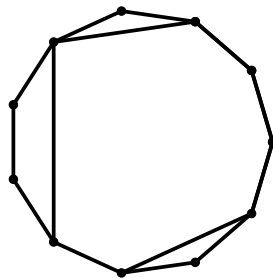


Figure 1: A cycle with three non-intersecting chords: two of length 2 and one of length 3

Predecessors in cycles. Some of our proofs use arguments involving rotations, and as such the following notation will be useful. Given a cycle C with an arbitrary direction, and a vertex u in C , we denote by u^- the predecessor of u in C (according to the arbitrary direction of C). More generally, we will write u^{-i} for the vertex in C obtained by taking i steps backwards from u . Additionally, given two vertices u, v in C , we write $C_{u \rightarrow v}$ for the subpath of C with ends u, v which contains the successor of u , and write $C_{u \leftarrow v}$ for the other subpath of C with ends u, v . Similarly, for a path P and vertices u, v in P , we write $P_{u \rightarrow v}$ for the subpath of P with ends u, v .

More path and cycle notation. For a path P we write $|P|$ for its *order*, namely its number of vertices, and $\ell(P)$ for its *length*, namely its number of edges. As the order and length of a cycle are the same, we normally stick to the more standard notation $|C|$ for its length. For distinct vertices v_1, \dots, v_ℓ , denote by $v_1 \dots v_\ell$ the path with edges $v_1v_2, \dots, v_{\ell-1}v_\ell$, and by $(v_1 \dots v_\ell)$ the cycle with edges $v_1v_2, \dots, v_{\ell-1}v_\ell, v_\ell v_1$. For vertices x, y , an xy -*path* is a path with ends x and y . If P is an xy -path and Q is a yz -path, and P and Q share only the vertex y , then $xPyQz$ denotes the path which is the concatenation of P and Q . Similarly, if P and Q are xy -paths that do not share vertices other than x and y , then $(xPyQx)$ is the cycle which is the union of P and Q . We sometimes use PQ and (PQ) for the same things if the ends are not explicitly known. We denote by P_k the path on k vertices.

3.1 Paths with many short chords

The next lemma finds a path with many short, non-intersecting chords.

Lemma 3.1. *Suppose that G is a graph satisfying $\delta(G) > \alpha(G)$. Then for every integer $k \leq \alpha(G)/6$, there is a path of length at most $3k$ which has k pairwise non-intersecting chords of length 2 or 3, at least one of which has length 2.*

Proof. Write $\alpha = \alpha(G)$. We define paths P_1, \dots, P_k such that the following holds for $i \in [k]$: $|P_i| \leq 3i$; and P_i has i pairwise non-intersecting chords of length 2 or 3, at least one of which with length 2. To start, notice that G contains a triangle (as the neighbourhood of any vertex has size at least $\alpha + 1$ and is thus not independent). Let x, y, z be the vertices of a triangle, define $P_1 = xyz$, and notice that xz is a chord of length 2 in P_1 . Now suppose that P_1, \dots, P_i are defined and $i \in [k - 1]$, and let x be one of P_i 's ends. Consider the graph $G' = G - (V(P_i) - \{x\})$. If x is in a triangle in G' , let y, z be the other two vertices in such a triangle, define $P_{i+1} = P_ixyz$, and observe that xz is a chord of length 2 that does not intersect the chords of P_i . Similarly, if x has a neighbour y which is in a triangle yzw in G' (with $z, w \neq x$), define $P_{i+1} = P_ixyzw$, and notice that yw is a chord of length 2. It remains to consider the case where x and one of its neighbours y are not in a triangle in G . Then the sets $N_x := N_{G'}(x) - \{y\}$ and $N_y := N_{G'}(y) - \{x\}$ are independent and disjoint. Notice that $|N_x|, |N_y| > \alpha - 3k \geq \alpha/2$. Thus $N_x \cup N_y$ is a set of size larger than α , showing that it contains an edge zw , which must have one end in N_x and the other in N_y ; say $z \in N_x$ and $w \in N_y$. Define $P_{i+1} = P_ixzwy$, and notice that xy is a chord of length 3 which does not intersect any chords in P_i . This completes the proof that a sequence P_1, \dots, P_k with the required properties exists. The path P_k satisfies the requirements of the lemma. \square

A path P as in Lemma 3.1 has the useful property that it can be shortened by any amount up to k . This is formalised in the next proposition.

Proposition 3.2. *Let P be a path with ends x, y and k non-intersecting chords of length 2 or 3, with at least one of length 2. Then, for every $k' \leq k$, there is an xy -path P' with $V(P') \subseteq V(P)$ and $|P'| = |P| - k'$.*

Proof. Consider a collection of k non-intersecting chords of length 2 or 3 in P , at least one of which having length 2, and denote the number of chords of length 2 in this collection by a and the number of chords of length 3 by b . We claim that there exist a', b' such that $0 \leq a' \leq a$, $0 \leq b' \leq b$, and $a' + 2b' = k'$. Indeed, taking $b' = \min\{\lfloor k'/2 \rfloor, b\}$ and $a' = k' - 2b'$ works. Let $e_1, \dots, e_{a'+b'}$ be non-intersecting chords, a' of which of length 2 and the rest of length 3, and consider the path P' obtained by replacing the subpath of P between the ends of e_i by e_i itself, for $i \in [a'+b']$. Then $|P'| = |P| - a' - 2b' = |P| - k'$. \square

3.2 Cycle-complete Ramsey numbers

We now mention two results regarding cycle-complete Ramsey numbers. These refer to the Ramsey number $r(C_\ell, K_s)$, namely the minimum n such that every n -vertex graph contains either a cycle of length ℓ or an independent set of size s . The first is an early result about these numbers, which tends to be useful when ℓ is small.

Theorem 3.3 (Erdős–Faudree–Rousseau–Schelp [16]). *Let $\ell \geq 3$ and $s \geq 2$, and write $x = \lfloor \frac{\ell-1}{2} \rfloor$. Then*

$$r(C_\ell, K_s) \leq \left((\ell - 2)(s^{1/x} + 2) + 1 \right) (s - 1).$$

The next result determines $r(C_\ell, K_s)$ precisely when ℓ is large with respect to s , and resolves a conjecture from [16].

Theorem 3.4 (Keevash–Long–Skokan [21]). *There is a constant $c \geq 1$ such that the following holds for $s \geq 3$ and $\ell \geq \frac{c \log s}{\log \log s}$.*

$$r(C_\ell, K_s) = (\ell - 1)(s - 1) + 1.$$

4 Upper range

Our aim in this section is to prove Lemma 2.1, restated here, which guarantees the existence of long cycles.

Lemma 2.1 (Upper range). *Let $0 < \delta \ll 1$ and let $n, \alpha, \ell \geq 1$ be integers, satisfying: α is sufficiently large, $\alpha \leq n \leq 4\alpha^2$, and $n/\delta\alpha \leq \ell \leq n$. Suppose that G is a graph on n vertices with $\alpha(G) = \alpha$ and $\kappa(G) > \alpha$. Then G has a cycle of length ℓ .*

This section is the one where our proof differs the most from [14]. Key new components here are a new rotation-extension technique, and the use of rotation-extension arguments given a cycle C for which there are no long paths outside of C .

The proof proceeds as follows. First, we apply Lemma 3.1 to find a path P_0 with many non-intersecting chords of length 2 or 3, which can be extended to a cycle of length at most ℓ . Using Lemma 4.1 below,

we show that there is a cycle C that either extends P_0 or contains many non-intersecting chords of length 2, which is not much longer than ℓ , such that either $G - V(C)$ is P_5 -free, or $|C| \geq \ell$. If $|C| < \ell$, then $G - V(C)$ is P_5 -free, and we use Lemma 4.2 to find a cycle of length at most ℓ . Otherwise, we use two lemmas from [12, 14] (Lemmas 4.3 and 4.4 below) to shorten C gradually to have length slightly more than ℓ , so that it can further be shortened to have length exactly ℓ using the short chords.

4.1 Lemmas

In this subsection we state the lemmas we will use in the proof of Lemma 2.1. Most of the work towards proving Lemma 2.1 will go into the proof of the following lemma.

Lemma 4.1. *Let $0 < \delta \ll 1$, and let $n, \alpha, \ell \geq 1$ be integers such that n is large.*

Let G be a graph on n vertices with $\alpha(G) = \alpha$ and $\kappa(G) > \alpha$. Let C_0 be a cycle of length at most ℓ , and let P_0 be a subpath of C_0 of length at most $\delta\alpha$. Then there is a cycle C satisfying the following three properties.

- (1) $|C| \leq \ell + \frac{n}{\delta\alpha}$.
- (2) Either $G - V(C)$ has no paths of length 4, or $|C| \geq \ell$.
- (3) Either $P_0 \subseteq C$, or C has at least $\delta\alpha$ pairwise non-intersecting chords of length 2.

Given a cycle as guaranteed by the last lemma, if its length is less than ℓ , then we can apply the following lemma to find a cycle of length exactly ℓ .

Lemma 4.2. *Let $\ell, n \geq 1$ be integers satisfying $3 \leq \ell \leq n$. Let G be a graph on n vertices, satisfying $\kappa(G) > \alpha(G)$. Suppose that C_0 is a cycle in G satisfying: $|C_0| < \ell$; there are at least 18 pairwise non-intersecting chords of length 2 or 3 in C_0 ; and $G - V(C_0)$ is P_5 -free. Then G has a cycle of length exactly ℓ .*

Otherwise, we apply one of the following results, due to Draganic–Munhá–Correia–Sudakov [12, 14], allowing us to shorten a given path by a relatively small amount.

Lemma 4.3 (Lemma 2.9 in [14] (a Consequence of Proposition 2.9 in [12])). *Let G be an n -vertex graph with independence number at most α , and let P be a path in G with ends x, y , satisfying $|P| > 4\alpha$. Then there is an xy -path P' such that $|P| - \left\lceil \frac{20\alpha^2}{|P|} \right\rceil \leq |P'| < |P|$.*

Lemma 4.4 (Lemma 2.8 in [14]). *Let G be an n -vertex graph with minimum degree δ , and let P be a path with ends x, y and length at least $20n/\delta$. Then there is an xy -path P' satisfying $|P| - 20n/\delta \leq |P'| < |P|$.*

4.2 Proof of upper range lemma

We now prove the upper range lemma, using the results mentioned in the previous subsection.

Proof of Lemma 2.1. Let η be a constant satisfying $\delta \ll \eta \ll 1$. Suppose that ℓ satisfies $\frac{n}{\delta\alpha} \leq \ell \leq n$ and write $k = \min\{\eta\alpha, \ell/6\}$. By Lemma 3.1, there is a path P_0 of length at most $3k$ which has k non-intersecting chords of length 2 or 3, with at least one of length 2. Let C_0 be a shortest cycle that extends P_0 .

We claim that $|C_0| \leq \ell$. Indeed, denote by x, y the ends of P_0 , and consider the graph G_0 , obtained from G by removing the interior vertices of P_0 . Then $\kappa(G_0) \geq \kappa(G) - 3k \geq \alpha/2$. Thus, there are $\alpha/2$ paths from x to y in G_0 with non-intersecting interiors. Let Q_0 be a shortest xy -path; then $|Q_0| \leq n/(\alpha/2) = 2n/\alpha \leq \ell/2$, implying that $|C_0| \leq \ell$, as claimed.

Let C_1 be a cycle as guaranteed by Lemma 4.1, applied with P_0 and C_0 and $\delta_{4.1} = \eta$. If $|C_1| < \ell$, then by (2), the graph $G - V(C_1)$ is P_5 -free. Additionally, by (3), by choice of P_0 , and by α being large, C_1 has 18 non-intersecting chords of length 2 or 3. Lemma 4.2 thus yields a cycle of length exactly ℓ , as required. So, we may assume that $|C_1| \geq \ell$.

Suppose now that $P_0 \subseteq C_1$, let P_1 be the subpath of C_1 obtained by removing the edges of P_0 , denote its ends by x, y , and let G_1 be the graph obtained from G by removing the interior vertices of P_0 . Then $\delta(G_1) \geq \alpha/2$ and $\alpha(G_1) \leq \alpha$. Write $\ell' = \ell - \ell(P_0)$. Let P_2 be the shortest xy -path of length at least ℓ' in G_1 .

Claim 4.5. $\ell(P_2) \leq \ell' + k$.

Proof. Suppose that $\ell(P_2) > \ell' + k$. By Lemma 4.4, there is an xy -path P'_2 in G_1 that satisfies $\ell(P_2) - r \leq \ell(P'_2) < \ell(P_2)$, where $r = \frac{20|G_1|}{\delta(G_1)} \leq \frac{40n}{\alpha}$. If $k \geq \frac{40n}{\alpha}$, then $\ell(P'_2) \geq \ell' + k - r \geq \ell'$, contradicting the minimality of P_2 .

We thus have $k \leq \frac{40n}{\alpha}$. In particular, $k = \eta\alpha$ (because $\ell/6 \geq \frac{40n}{\alpha}$, $\alpha^2 \leq \frac{40n}{\eta}$, and $\frac{\ell}{\alpha} \geq \frac{n}{\delta\alpha^2} \geq \frac{\eta}{40\delta} \geq 10$ (by $\delta \ll \eta$). Note that $\ell(P_2) \geq \ell - \ell(P_0) \geq \ell/2 \geq 5\alpha$ (using the last inequality). Hence, by Lemma 4.3, there exists an xy -path P'_2 in G_1 satisfying $\ell(P_2) - r \leq \ell(P'_2) < \ell(P_2)$, where $r = \left\lceil \frac{20\alpha^2}{|P_2|} \right\rceil \leq \frac{20\alpha^2}{\ell/2} + 1 = \frac{40\alpha^2}{\ell} + 1 \leq \frac{1600\delta\alpha}{\eta} + 1 \leq \eta\alpha = k$ (using $\frac{\ell}{\alpha} \geq \frac{\eta}{40\delta}$, $\delta \ll \eta$, and α being large). Thus $\ell(P'_2) \geq \ell(P_2) - r \geq \ell'$, again contradicting the minimality of P_2 . \square

Recall that P_0 has k non-intersecting chords of length 2 or 3, with at least one of length 2. Then, by Proposition 3.2, there is an xy -path P'_0 , with $V(P'_0) \subseteq V(P_0)$ and $\ell(P'_0) = \ell - \ell(P_2)$ (indeed, we have $\ell(P_0) - k \leq \ell - \ell(P_2) \leq \ell(P_0)$ by the above claim and choice of P_2). The concatenation of P'_0 and P_2 is a cycle of length ℓ , as required.

It remains to consider the case $P_0 \not\subseteq C_1$. Then C_1 has at least $\eta\alpha$ non-intersecting chords of length 2, and $|C_1| \leq \ell + \frac{n}{\eta\alpha}$.

Suppose first that $n \leq \eta^2\alpha^2$. Write $t = |C_1| - \ell$; then $t \leq \eta\alpha$ using the last condition above. Pick t non-intersecting chords of length 2 in C_1 , and, for each of these chords e , replace the subpath of length 2 of C_1 between the vertices of e by e itself. This yields a cycle of length exactly ℓ .

Now consider the case $n \geq \eta^2\alpha^2$. Let P_1 be a subpath of C_1 of length $\ell/2$ with a maximum number of non-intersecting chords of length 2, and let P_2 be the subpath of C obtained by removing the edges of P_1 . Noting that $|C| \leq \ell + \frac{n}{\eta\alpha} \leq 2\ell$, we conclude that P_1 has at least $\eta\alpha/4$ non-intersecting chords of length 2. Denote the ends of P_2 by x, y , and let P_3 be a shortest xy -path in $G[V(P_2)]$ of length

at least $\ell/2$. We claim that $\ell(P_3) \leq \ell/2 + \eta\alpha/2$. If not, then by Lemma 4.3 there is an xy -path P'_3 in $G[V(P_3)]$ with $\ell(P_3) - r \leq \ell(P'_3) < \ell(P_3)$, where $r = \left\lceil \frac{20\alpha^2}{\ell} \right\rceil \leq \frac{20n}{\eta^2\ell} + 1 \leq \frac{20\delta\alpha}{\eta^2} + 1 \leq \eta\alpha/4$ (using $\ell \geq n/\delta\alpha$, $\eta \gg \delta$, and α being large), a contradiction to the minimality of P_3 . To finish, note that there is an xy -path P'_1 in $G[V(P_1)]$ of length exactly $\ell - \ell(P_3)$. Indeed, since $\ell(P_1) = \ell/2$, we have $\ell(P_1) - \eta\alpha/4 \leq \ell - \ell(P_3) \leq \ell(P_1)$, and P_1 has at least $\eta\alpha/4$ non-intersecting chords of length 2, so we can use some of these chords to shorten P_1 as needed. Concatenating P'_1 and P_3 yields a cycle of the desired length. \square

4.3 Finding a cycle with many chords

In this subsection we prove Lemma 4.1, the main step towards Lemma 2.1. The proof proceeds roughly as follows. We take C to be a longest cycle of length at most which satisfies (1) and (3). Our task is to show that (2) also holds, so suppose not. This means that $|C| < \ell$ and there is a components H in $G - V(C)$ that contains a path of length at least 4. To reach a contradiction, we will show that there is a cycle C' satisfying: $|C'| > |C|$; the vertices outside of C in C' are in H and form a subpath of C' ; and C' contains either P_0 or many chords of length 2. For this to be a contradiction, we want $|P'| \leq \frac{n}{\delta\alpha}$, due to (1). This will be achieved in the first two claims, Claims 4.6 and 4.7, the latter asserting that if x, y can be joined by a path of length at least 4 in H , then there is such a path of length at most $\frac{n}{\delta\alpha}$. It is easy to conclude that $|C| > \alpha$ (see Claim 4.8). Now, a standard rotation technique yields a cycle C' such that: $|C'| > |C|$; C' misses at most two edges of C ; and the vertices not in C' induce a subpath of C' . This implies the desired contradiction if C has at least $\delta\alpha + 2$ chords, so we may assume this is not the case (see Claim 4.9). A somewhat more intricate rotation argument (which was outlined in the introduction) (which was outlined in the introduction) (which was outlined in the introduction) implies that H has no path of length at least 4 whose ends have relatively small degree in H (see Claim 4.11). We then use structural arguments to find a relatively large subset $A \subseteq V(H)$ whose every two vertices can be joined by a path of length at least 4 in $H[A]$, and whose almost every vertex has few neighbours in $H - A$. A similar rotation argument now yields the desired contradiction, using the properties of A .

Proof of Lemma 4.1. Let η be a constant satisfying $\delta \ll \eta \ll 1$. Let C be a longest cycle satisfying (1) and (3), such that if $P_0 \not\subseteq C$ then $|C| > \alpha$. We will show that it also satisfies (2). Suppose not, then $|C| < \ell$ and $G - V(C)$ contains a path of length 4. Let H be a component in $G - V(C)$ that contains such a path.

Define E as follows. If $P_0 \subseteq C$, set $E = E(P_0)$. Otherwise, let E be a set of $2\delta\alpha$ edges in C corresponding to $\delta\alpha$ pairwise non-intersecting chords of length 2 in C .

The next claim is in preparation for the claim after that, which says that any two vertices in H which are joined by a path of length at least 4 in H , are joined by such a path which is not too long. This will help us control the length by which the cycle C is extended in various situations, allowing us to satisfy (1).

Claim 4.6. *There is no path P in H with $\frac{n}{\eta^3\alpha} \leq |P| \leq \frac{n}{\delta\alpha}$ whose ends have degree at most $(\frac{1}{2} - \eta)\alpha$ in H .*

Proof. Suppose that P is such a path and denote its ends by x, y . Let X be the set of neighbours of x in C that are not incident to edges of E ; then $|X| \geq d_G(x) - d_H(x) - 2|E| \geq \alpha - (\frac{1}{2} - \eta)\alpha - 4\delta\alpha \geq (\frac{1}{2} + \eta^2)\alpha$. Define Y similarly with respect to y , then similarly $|Y| \geq (\frac{1}{2} + \eta^2)\alpha$.

Write $X^- = \{x^- : x \in X\}$ and define Y^- analogously for y . We claim that $X^- \cup Y^-$ is independent. Indeed, suppose not, and pick $u, v \in X \cup Y$ such that u^-v^- is an edge. If $u, v \in X$, define

$$C' = (uC_{u \rightarrow v}v^-u^-C_{u^- \leftarrow v}vxu). \quad (1)$$

(See Figure 2, and note that this make sense also when $u^- = v$.) Moreover, $|C'| = |C| + 1 \leq \ell$ and

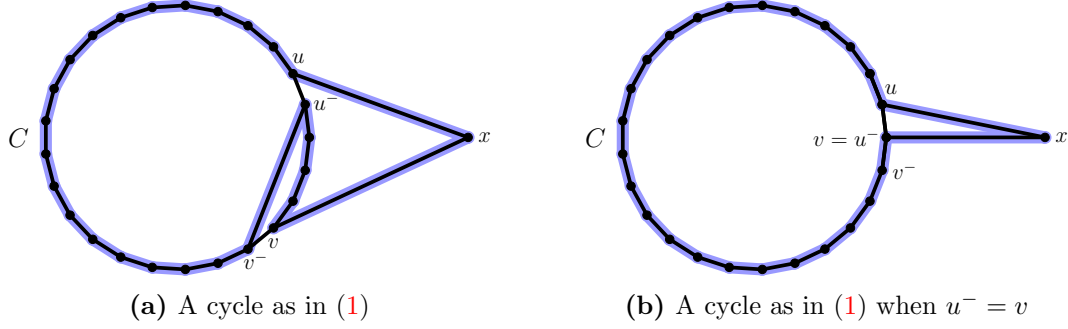


Figure 2: Cycles as in (1)

$E \subseteq C'$, so (1) and (3) both hold, a contradiction to the maximality of C . The same argument works if $u, v \in Y$, so suppose now that $v \in X$ and $u \in Y$, and define

$$C' = (uC_{u \rightarrow v}v^-u^-C_{u^- \leftarrow v}vxPyu). \quad (2)$$

(See Figure 3, and note that this also works when $v = u^-$.)

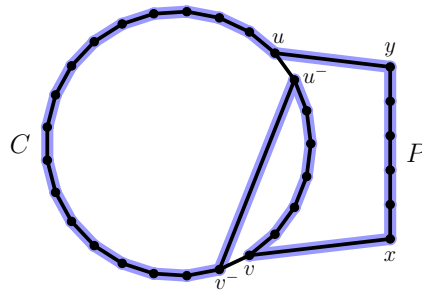


Figure 3: A cycle as in (2)

Then $|C'| \leq |C| + |P| \leq |C| + \frac{n}{\delta\alpha}$, contradicting the maximality of C .

Since $X^- \cup Y^-$ is independent, it has size at most α , showing that $|X^- \cap Y^-| \geq \eta^2\alpha$, and thus $|X \cap Y| \geq \eta^2\alpha$. Consider the segments of C between consecutive elements of $X \cap Y$. At most $2\delta\alpha$ of them contain an edge of E , leaving at least $\eta^3\alpha$ segments without any edges of E . Pick a segment I of

length at most $\frac{|C|}{\eta^3\alpha} \leq \frac{n}{\eta^3\alpha}$, and denote its ends by u and v . Define

$$C' = (vC_{v \leftarrow u}uyP xv), \quad (3)$$

(see Figure 4). Then $|C'| \geq |C| + |P| - (|I| - 2) \geq |C| + 1 > |C|$ (using $|P| \geq \frac{n}{\eta^3\alpha} \geq \ell(I) = |I| - 1$) and $|C'| \leq |C| + |P| \leq |C| + \frac{n}{\delta\alpha}$. This contradicts the maximality of C . \square

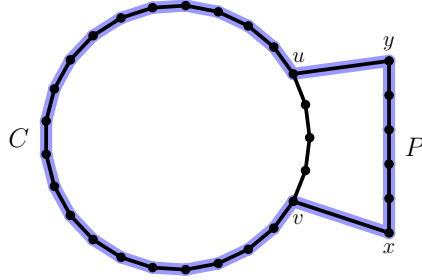


Figure 4: A cycle as in (3)

Claim 4.7. *If x, y are vertices in H that are joined by a path of length at least 4, then there is such a path P in H with $|P| \leq \frac{n}{\delta\alpha}$.*

Proof. Let $x, y \in V(H)$ which can be joined by a path of length at least 4 in H . Let P be a shortest xy -path in H of length at least 4. We need to show $|P| \leq \frac{n}{\delta\alpha}$. Suppose this is not the case.

Denote the set of first and last four vertices in P by X , let U be a maximal set of vertices in $P - X$, that are pairwise at distance at least 3 from each other on P ; so $|U| \geq \lfloor \frac{|P|-8}{3} \rfloor \geq \frac{|P|}{4} \geq \frac{n}{5\delta\alpha}$. Let W be the set of vertices in U with degree at least $\alpha/4$ in H . Then the sets $N_H(w) - X$, with $w \in W$, are pairwise disjoint sets (due to the minimality of P) of size at least $\alpha/4 - 8 \geq \alpha/5$. It follows that $|W| \leq \frac{n}{\alpha/5} \leq |U|/2$.

By the previous paragraph, at least $|U|/2 \geq |P|/8$ of the vertices in P have degree at most $\alpha/4$ in H . Since P can be divided into at most $\frac{|P|}{n/2\delta\alpha}$ paths on at most $\frac{n}{\delta\alpha}$ vertices (by taking as many paths as possible of order exactly $\frac{n}{2\delta\alpha}$ and adding the remainder to the last path), there exists a subpath Q of order at most $\frac{n}{\delta\alpha}$ that contains at least $\frac{|P|/8}{|P|/(n/2\delta\alpha)} = \frac{n}{16\delta\alpha}$ vertices of degree at most $\alpha/4$ in H . Then Q in turn contains a subpath of order between $\frac{n}{16\delta\alpha}$ and $\frac{n}{\delta\alpha}$ whose ends have degree at most $\alpha/4 \leq (\frac{1}{2} - \eta)\alpha$ in H , contradicting Claim 4.6. \square

For technical reasons, it will be useful to know that $|C| > \alpha$. This is proved in the following claim.

Claim 4.8. $|C| > \alpha$.

Proof. Suppose not. Then, by connectivity, every vertex in C has a neighbour in H . Notice that C has two consecutive vertices u, v that are not the ends of the same edge in E ; this is because, by $|C| \leq \alpha$ we have $P_0 \subseteq C$ and so E is the edge set of P_0 . Let x be a neighbour of v in H , let y be a neighbour of u in H , and let P be a shortest xy -path in H (if $x = y$ then P is a single vertex). Then $|P| \leq \frac{n}{\delta\alpha}$ by Claim 4.7. A cycle as in (3) yields a contradiction to the maximality of C . \square

Claim 4.9. *There is no collection of at least $\delta\alpha + 2$ non-intersecting chords in C of length 2.*

Proof. Suppose that C has at least $\delta\alpha + 2$ non-intersecting chords of length 2. Let U be the set of vertices in C with neighbours in H . Then, because $\kappa(G), |C| > \alpha$ (by assumption and Claim 4.8), we have $|U| > \alpha$, implying that there are $u, v \in U$ such that u^-v^- is an edge. Let x, y be neighbours of v, u in H , and let P be a shortest path in H between u and v . Define C' as in (2) (see Figure 3). Then $|C'| > |C| > \alpha$ and $|C'| \leq |C| + |P| \leq \ell + \frac{n}{\delta\alpha}$, by Claim 4.7. Moreover, since C' misses at most two edges of C , it has at least $\delta\alpha$ non-intersecting chords of length 2. This contradicts the maximality of C . \square

Claim 4.10. *There is no path of length at least 4 in H whose ends have degree at most $(\frac{1}{2} - \eta)\alpha$ in H .*

Proof. Suppose that P is a path of length at least 4 whose ends, denoted x and y , have degree at most $(\frac{1}{2} - \eta)\alpha$ in H . By Claim 4.7, we may assume $|P| \leq \frac{n}{\delta\alpha}$.

Let X be the set of vertices in C that are neighbours of x but not of y , and are at distance at least 4 on C from any edge in E . Define Y similarly with the roles of x and y reversed, and let Z be the set of vertices in C that are neighbours of both x and y , and are at distance at least 4 from any edge in E . Then $|X \cup Z| \geq \alpha - d_H(x) - 10|E| \geq (\frac{1}{2} + \eta^2)\alpha$, with the same estimate holding for $|Y \cup Z|$. Define

$$X^- = \{x^- : x \in X\} \quad Y^- = \{y^- : y \in Y\} \quad Z^- = \{z^- : z \in Z\} \quad Z^{-3} = \{z^{-3} : z \in Z\}.$$

We claim that these four sets are pairwise disjoint. By disjointness of X, Y, Z , if not, then $Z^{-3} \cap (X \cup Y \cup Z) \neq \emptyset$, showing that there are vertices u, v at distance 2 on C such that v is a neighbour of x and u a neighbour of y . This is a contradiction, as can be seen by defining C' as in (3) (see Figure 4). Indeed, then $|C'| = |C| - 1 + |P| > |C|$ (using $|P| \geq 5$), and $|C'| \leq |C| + |P| \leq |C| + \frac{n}{\delta\alpha}$. So, the sets X^-, Y^-, Z^-, Z^{-3} are pairwise disjoint, showing that $|X^- \cup Y^- \cup Z^- \cup Z^{-3}| = |X \cup Z| + |Y \cup Z| \geq (1 + 2\eta^2)\alpha$. Since $\alpha(G) = \alpha$, there is a matching M of size at least $\eta^2\alpha$ in $X^- \cup Y^- \cup Z^- \cup Z^{-3}$.

Let e be an edge in M . We claim that it is of form z^-z^{-3} , for some $z \in Z$. Indeed, suppose not. Then one of the following holds: there exist $u, v \in X \cup Y \cup Z$ such that $e = u^-v^-$; there exist $u \in X \cup Y \cup Z$ and $v \in Z$ such that $e = u^-v^{-3}$; and there exist $u, v \in Z$ such that $e = u^{-3}v^{-3}$.

We show now that each of these cases leads to a contradiction. This can be seen in the first case via a cycle as in (1) or (2). In the second case, by symmetry, we may assume that u is a neighbour of y . Consider the cycle

$$C' = (uC_{u \rightarrow v^{-3}}v^{-3}u^-C_{u^- \leftarrow v}v x P y u). \quad (4)$$

(See Figure 5a.) Then, $|C'| = |C| - 2 + |P| > |C|$, and $|C'| \leq |C| + |P| \leq |C| + \frac{n}{\delta\alpha}$. (Here we are implicitly assuming that u, v are at distance at least 4 on C ; if this is not the case then we can reach a contradiction via a cycle of form (3).) Finally, in the third case we can similarly reach a contradiction via the cycle

$$C' = (uC_{u \rightarrow v^{-3}}v^{-3}u^{-3}C_{u^{-3} \leftarrow v}v x P y u). \quad (5)$$

(See Figure 5b.)

The previous paragraph implies that every edge in M is of the form z^-z^{-3} for some $z \in Z$. In particular, M is a set of $\eta^2\alpha \geq \delta\alpha + 2$ pairwise non-intersecting chords of length 2 in C , contradicting Claim 4.9. \square

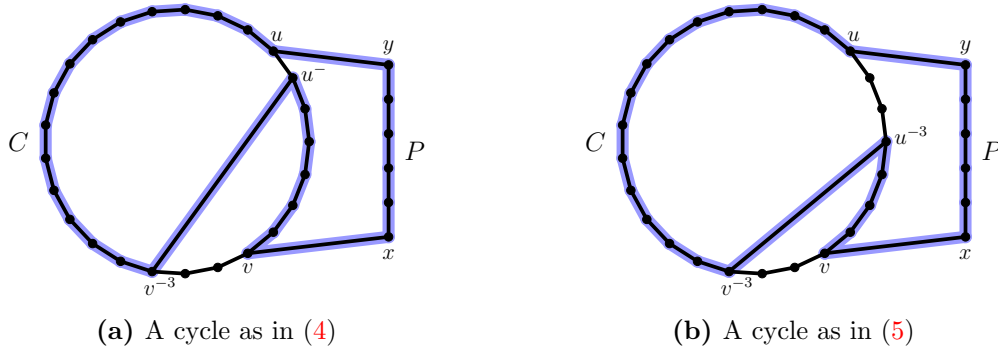


Figure 5: Cycles as in (4) and (5)

Consider a longest path P in H . Then $\ell(P) \geq 4$, by assumption on H . By the last claim, at least one of P 's ends has degree at least $(\frac{1}{2} - \eta)\alpha$ in H ; say x is such an end. Then all of x 's neighbours in H are in $V(P)$, by maximality of P , showing that there is a cycle C' in H of length at least $(\frac{1}{2} - \eta)\alpha$. Notice that any two vertices in C' can be joined by a path of length at least 4 in C' . Thus we can take A to be a largest set satisfying $V(C') \subseteq A \subseteq V(H)$, such that for any two vertices in A there is a path of length at least 4 in $H[A]$ that joins these two vertices.

Claim 4.11. *There is a subset $A' \subseteq A$ of size at least $|A| - \delta^{-3}$ such that every $a \in A'$ has at most $\delta\alpha$ neighbours in $V(H) - A$.*

Proof. We claim that there is no path P in H of length at least 2, whose ends are in A but whose interior is outside of A . Indeed, otherwise, take $A' = A \cup V(P)$. It is easy to check that every two vertices $x, y \in A'$ can be joined by a path of length at least 4 in $H[A']$. Indeed, this holds by choice of A if $x, y \in A$, so suppose that x is in the interior of P . If $y \in A$, let z be an end of P which is not y . Then concatenating a yz -path of length at least 4 in $H[A]$ with the segment $P_{z \rightarrow x}$, yields a path with the required properties. Similarly, if x, y are both in the interior of P , let z, w be the ends of P such that z is closer to x than to y in P . Now concatenate $P_{x \rightarrow z}$ with a zw -path of length at least 4 in $H[A]$, and with $P_{w \rightarrow y}$, to get a path as desired.

Similarly, there is no cycle of length at least 8 that intersects A in exactly one vertex.

By the first paragraph, for every $x \in V(H) - A$ there is a vertex $a(x) \in A$ such that $a(x)$ disconnects x from A in H . Let $X(a)$ be the set of vertices $x \in V(H) - A$ such that $a(x) = a$. Notice that, by Claim 4.10 and the choice of A , for at most one $a \in A$, the set $X(a)$ contains a vertex of degree less than $(\frac{1}{2} - \eta)\alpha$ in H . Let a' be this vertex (if exists).

Let $\mathcal{C}(a)$ be a maximal collection of pairwise vertex-disjoint cycles of length at least $\delta\alpha$ in $H[X(a) - \{a\}]$, for $a \in A - \{a'\}$. Let A' be the set of vertices $a \in A - \{a'\}$ such that $|\mathcal{C}(a)| \leq \delta\alpha$. Then $|A - A'| \leq \frac{n}{(\delta\alpha)^2} + 1 \leq \frac{4}{\delta^2} + 1 \leq \delta^{-3}$, by the assumption $n \leq 4\alpha^2$, and by disjointness of the cycles in $\mathcal{C}(a)$ and of the sets $X(a)$.

It remains to show that every $a \in A'$ has at most $\delta\alpha$ neighbours in $V(H) - A$. Indeed, let $a \in A'$, and write $\mathcal{C}(a) = \{C_1, \dots, C_r\}$. Notice that there are no two vertex-disjoint (except at a) paths from a to C_i , for any $i \in [r]$, because otherwise the second paragraph would be violated. There is therefore a vertex y_i that disconnects a from C_i , for $i \in [r]$. If a has more than r neighbours in $V(H) - A$, let x be such a

neighbour which is not in $\{y_1, \dots, y_r\}$. Then the connected component H' of $H[X(a) - (\{a, y_1, \dots, y_r\})]$ that contains x is a non-empty graph with minimum degree at least $(\frac{1}{2} - \eta)\alpha - (r + 1) \geq \delta\alpha$. So H' has a cycle C' of length at least $\delta\alpha$. Notice that C' is vertex-disjoint of the cycles C_1, \dots, C_r , contradicting the maximality of $\mathcal{C}(a)$. This shows that a has at most $r \leq \delta\alpha$ neighbours in $H - A$, for every $a \in A'$. \square

Let W be the set of vertices in C with at least two neighbours in A . We claim that $|A| + |W| \geq (\frac{1}{2} + \eta)\alpha$. Indeed, this is clearly the case if $|A| \geq (\frac{1}{2} + \eta)\alpha$, so suppose otherwise. Let U be the set of vertices in C that are neighbours of vertices in A , and let U' be the set of vertices in U which are not in edges in E . Notice that $(U')^-$ is independent, because otherwise a cycle as in (1) or (2), together with Claim 4.7, would lead to a contradiction. Thus $|U'| \leq \alpha$ and $|U| \leq |U'| + 2|E| \leq (1 + 2\delta)\alpha$. We conclude that $|W| \geq \alpha/3$, say. Indeed, otherwise each vertex in A' contributes at least $(\frac{1}{2} - \eta - \delta - \frac{1}{3})\alpha \geq \frac{\alpha}{7}$ distinct vertices to U , resulting in U having size at least $|A'| \cdot \frac{\alpha}{7} \geq \frac{\alpha^2}{28} > (1 + 2\delta)\alpha$, a contradiction. It follows that $|A| + |W| \geq (\frac{1}{2} - \eta + \frac{1}{3})\alpha \geq (\frac{1}{2} + \eta)\alpha$, as claimed (using $|A| \geq (\frac{1}{2} - \eta)\alpha$).

By connectivity and by $|C| > \alpha$, there are at least $\min\{|A|, \alpha - |W|\}$ pairwise vertex-disjoint paths from A to $C - W$; denote such a collection of paths by \mathcal{P} and assume that interiors of paths in \mathcal{P} are disjoint of C and A . Let X be the set of vertices in C that are either in W or an end of a path in \mathcal{P} , and moreover are at distance at least 4 from E . Then $|X| \geq \min\{|A| + |W|, \alpha\} - 8\delta\alpha \geq (\frac{1}{2} + \eta^2)\alpha$.

Claim 4.12. *Let $u, v \in X$ be distinct. Then there are distinct vertices x, y in H such that x is a neighbour of v and y of u , and there is an xy -path of length at least 4 in H .*

Proof. By symmetry, there are three cases to consider: $u, v \in W$; $v \notin W, u \in W$; and $u, v \notin W$. In the first case, there are distinct x, y in A such that x is a neighbour of v and y of u . By choice of A , there is an xy -path of length at least 4 in H . In the second case, let P be any path from v to A , whose interior is disjoint of C and A , and denote its end which is not v by x' . Let $y \in A$ be a neighbour of u which is not x , and let Q be an $x'y$ -path of length at least 4 in $H[A]$. Taking x to be the neighbour of v in P , the path PQ with v removed is an xy -path of length at least 4. Finally, in the third case, we may pick two vertex-disjoint paths P_1, P_2 from v, u to C , with interior disjoint of C and A . Let x', y' be the ends of P_1, P_2 in A , respectively, let x, y the neighbours of v, u in P_1, P_2 , and let Q be an $x'y'$ -path of length at least 4 in A . Then $P_1QP_2 - \{u, v\}$ is an xy -path of length at least 4. \square

One can now show that any two vertices in X are at distance at least 4 in C , via a cycle as in (3), using Claims 4.7 and 4.12.

In particular, the sets X^- and X^{-3} , defined as usual, are disjoint. Thus $|X^- \cup X^{-3}| \geq (1 + 2\eta^2)\alpha$, showing that there is a matching M in $X^- \cup X^{-3}$ of size at least $\eta^2\alpha$.

We claim that all edges in M are of form u^-u^{-3} , for some $u \in X$. Indeed, if not, there are distinct $u, v \in X$ such that one of $u^-v^-, u^-v^{-3}, u^{-3}, v^{-3}$ is an edge in M . Each of these cases leads to a contradiction, by considering cycles as in (2), (4), (5), using Claims 4.7 and 4.12 to find an appropriate P with $5 \leq |P| \leq \frac{n}{\delta\alpha}$.

Thus M is a matching of at least $\eta^2\alpha$ non-intersecting chords of length 2, a contradiction to Claim 4.9. This concludes the proof that $G - V(C)$ has no paths of length 4, so C satisfies the requirements of the lemma. \square

4.4 Extending a cycle whose complement is P_5 -free

In this subsection we prove Lemma 4.2. We first sketch its proof. Suppose that C is a cycle in a graph G satisfying: $|C| < \ell$; $G - V(C)$ is P_5 -free; and C has 18 non-intersecting chords of length 2 or 3. A simple rotation-extension argument (see Proposition 4.13) implies that every cycle C' containing the vertices of C can be extended by at most four vertices, from any specified component of $G - V(C')$. This alone allows us to find a cycle of length between $\ell - 3$ and ℓ . Another consequence of Proposition 4.13 is that, if for such a cycle C' there is a vertex outside of C' with at most one neighbour outside of C' , then C' can be extended by exactly that one vertex. It thus suffices to find a cycle C' such that: $V(C) \subseteq V(C')$; $|C'| \leq \ell$; and $G - V(C')$ has at least three vertices in components which are trees. This can be done by three applications of Proposition 4.15, asserting that a cycle C' containing the vertices of C can be extended by at most four vertices from a specified component H of $G - V(C')$, such that the remainder of the component is a non-empty tree. This might fail if $|C|$ is too close to ℓ , in which case we can use the short chords in C to get a cycle of length ℓ .

Before turning to the proof, we state and prove three propositions. The first one is a consequence of a simple rotation and extension operation.

Proposition 4.13. *Let G be a graph satisfying $\kappa(G) > \alpha(G)$. Let C be a cycle, H a component in $G - V(C)$, and u, v distinct vertices in H . Then there is a cycle C' such that: $V(C')$ is the union of $V(C)$ with the vertices of a path in H that contains u and avoids v ; and C' contains all but at most two edges of C .*

Proof. Write $\alpha = \alpha(G)$. Because $\kappa(G) > \alpha$, the graph $G - \{v\}$ is α -connected. If there is a vertex w in C such that w and w^- can be joined by a path in $G - \{v\}$, whose interior contains u and is disjoint of C , then we can define

$$C' = (wC_{w \rightarrow w^-} w^- Pw).$$

Then C' contains all edges in C except for ww^- and its vertex set is the union of $V(C)$ with the vertices of the path $P - \{w, w^-\}$. Suppose now that no such vertex w exists.

If $|C| < \alpha$, then by connectivity there are $|C|$ pairwise vertex-disjoint (except at u) paths from u to C in $G - \{v\}$, leading to a contradiction to the assumption we just made. So $|C| \geq \alpha$, implying that there are pairwise vertex-disjoint (except at u) paths P_1, \dots, P_α from u to C in $G - \{v\}$. We assume that the interiors of the paths P_i are disjoint of C . Let w_i be the end of P_i which is not u . Then by the above assumption no two vertices in $\{w_1, \dots, w_\alpha\}$ are consecutive, and there is no edge between u and w_i^- for any $i \in [\alpha]$.

The set $\{w_i^- : i \in [\alpha]\} \cup \{u\}$ has size $\alpha + 1$, and so it spans an edge e . Since there is no edge of form uw_i^- , we have $e = w_i^- w_j^-$ for some distinct $i, j \in [\alpha]$. We then get the cycle

$$C' = (uP_i w_i C_{w_i \rightarrow w_j^-} w_j^- w_i^- C_{w_i^- \leftarrow w_j} w_j P_j u),$$

like in (2) or Figure 3 but with different notation. It is easy to check that C' satisfies the requirements of the proposition. Indeed, C' uses all edges of C except for $w_i w_i^-$ and $w_j w_j^-$, and $V(C')$ is the union of $V(C)$ with the vertices of the path $(P_i \cup P_j) - \{w_i, w_j\}$. \square

The next proposition gives some structural information regarding P_5 -free graphs.

Proposition 4.14. *Let H be a connected P_5 -free graph. Then one of the following holds: H is a complete graph on four vertices; H is a tree; or there is a vertex u such that $H - \{u\}$ is a forest.*

Proof. Suppose that H is not a tree. Then it contains a cycle C . By P_5 -freeness, C has length 3 or 4. If it has length 4, then H has exactly four vertices, as adding a pendant edge to a 4-cycle results in the existence of a P_5 . In this case, it is easy to check that H is either complete, or there is a vertex whose removal from H leaves a tree. Suppose now that C is a triangle and that G has no 4-cycles. The latter assumption implies that no two vertices in C have a common neighbour outside of C . Hence, P_5 -freeness implies that there is at most one vertex in C , say u , that has neighbours outside of C , and that any neighbour of u outside of C has no neighbours other than u . In particular, $H - \{u\}$ is a forest. \square

The final proposition is a consequence of the former two.

Proposition 4.15. *Let G be a graph satisfying $\kappa(G) > \alpha(G)$. Let C be a cycle in a graph G and let H be a (non-empty) component in $G - V(C)$. Suppose that H is P_5 -free. Then there is a cycle C' such that: $|C| \leq |C'| \leq |C| + 4$; $V(C) \subseteq V(C')$; C' misses at most four edges of C ; and $H - V(C')$ is a non-empty forest.*

Proof. If H is a tree, there is nothing to prove (take $C' = C$). If there is a vertex u in H such that $H - \{u\}$ is a forest, then pick any $v \in V(H) - \{u\}$, and let C' be a cycle as guaranteed by Proposition 4.13. Then $|C| < |C'| \leq |C| + 4$ (the upper bound is by P_5 -freeness), $H - V(C')$ is a forest that contains v , and C avoids at most two edges of C .

It remains to consider the case $H \cong K_4$. By Proposition 4.13, there is a cycle C'' such that $V(C) \subsetneq V(C'') \subsetneq V(C) \cup V(H)$, and C'' avoids at most two edges of C . Let $H' = H - V(C'')$. If H' consists of at most two vertices, we are done. Otherwise, H' is a triangle. Then we can apply Proposition 4.13 again to obtain a cycle C' such that $V(C'') \subsetneq V(C') \subsetneq V(C'') \cup V(H')$ and C' misses at most two edges of C'' , and thus at most four edges of C . Now $H - V(C')$ has at least one and at most two vertices, and so it is a tree. \square

Finally, here is the proof of the lemma.

Proof of Lemma 4.2. Write $\alpha = \alpha(G)$.

First, we pick cycles C_1, C_2, C_3 as follows, where F_i is the union of components of $G - V(C_i)$ which are trees, and $G_i = G - V(F_i)$. If $|F_{i-1}| \geq i$ or $|C_{i-1}| \geq \ell - |F_{i-1}|$, define $C_i = C_{i-1}$. Otherwise, we can apply Proposition 4.15, with the cycle C_{i-1} and any component in $G_{i-1} - C_{i-1}$, to find a cycle C_i such that: $|C_{i-1}| \leq |C_i| \leq |C_{i-1}| + 4$; $V(C_{i-1}) \subseteq V(C_i)$; C_i misses at most four edges of C_{i-1} ; and $G_{i-1} - V(C_i)$ has a component which is a tree.

Notice that $|C_i| \leq \min\{|C_0| + 4i, \ell + 3\}$; $|F_i| \geq \min\{i, \ell - |C_i|\}$; and C_i avoids at most $4i$ edges of C_0 , for $i \in [3]$.

We distinguish three cases.

- $|C_3| = \ell + 2$. By the assumption that C_0 has at least 18 pairwise non-intersecting chords of length 2 or 3, and the fact that C_3 misses at most 12 edges of C_0 , the cycle C_3 has either a chord of length 3, or two non-intersecting chords of length 2. Both scenarios allow for shortening C_3 by exactly 2, to obtain an ℓ -cycle.
- $|C_3| \in \{\ell + 1, \ell + 3\}$. Notice that $|F_3| \geq 1$ (otherwise $C_3 = C_0$, a contradiction to $|C_0| < \ell$); let u be a vertex of degree at most 1 in F_3 . By Proposition 4.13, there is a cycle C_4 with $V(C_4) = V(C_3) \cup \{u\}$ which misses at most two edges of C_3 , and thus at most 14 edges of C . So $|C_4| \in \{\ell + 2, \ell + 4\}$ and C_4 has four non-intersecting chords of length 2 or 3. These can be used to form a cycle of length ℓ .
- $|C_3| \leq \ell$. Let C_4 be a longest cycle satisfying $V(C_3) \subseteq V(C_4) \subseteq V(G) - V(F_3)$ and $|C_4| \leq \ell$. We claim that $|C_4| \geq \ell - 3$ or $V(C_4) = V(G) - V(F_3)$. Indeed, suppose not. Then we can apply Proposition 4.13 with a vertex in $V(G) - (V(F_3) \cup V(C_4))$ to obtain a cycle C_5 with $V(C_4) \subseteq V(C_5) \subseteq V(G) - V(F_3)$ and $|C_5| \leq |C_4| + 4 \leq \ell$, contradicting the maximality of C_4 .
Write $r = \ell - |C_4|$. Recalling that $|F_3| \geq \min\{3, \ell - |C_3|\}$, we have $|F_3| \geq r$. As F_3 is a forest, there are distinct vertices u_1, \dots, u_r in F_3 such that u_j is a vertex of degree at most 1 in $F_3 - \{u_1, \dots, u_{j-1}\}$. Apply Proposition 4.13 r times, with u_j and its unique neighbour in $F_3 - \{u_1, \dots, u_{j-1}\}$ (if exists) in the j^{th} application. These applications result in a cycle C' with $V(C') = V(C) \cup \{u_1, \dots, u_j\}$. In particular $|C'| = \ell$.

This completes the proof that G contains a cycle of length ℓ . □

5 Middle range

In this section we prove Lemma 2.2 (restated here), about cycles whose length is not too small and not too large.

Lemma 2.2 (Middle range). *Let $0 < \delta \ll 1$ and let $n, \alpha, \ell \geq 1$ be integers, satisfying: α is sufficiently large, $\sqrt{n}/2 \leq \alpha \leq \delta n^{2/3}$, and $n/\alpha \leq \ell \leq \delta(n/\alpha)^2$. Suppose that G is an n -vertex graph with $\alpha(G) = \alpha$ and $\kappa(G) \geq \alpha$. Then G contains a cycle of length ℓ .*

The general outline of the proof is similar to the proof of the middle range in [14]. We start with a relatively short path P , which has the property that it can be shortened by any small amount (via short non-intersecting chords). We then extend P to a cycle C of length at most n/α . In [14] it was essentially trivial to show that such a cycle C exists. Here we need to work quite a lot harder; see Lemma 5.1.

Next, writing $P' = C - E(P)$, we show that there is another path P'' with the same ends as P' , such that together with P it forms a cycle of length slightly more than ℓ . This is achieved using Lemma 5.2, which strengthens Lemma 2.10 in [14].

Finally, this can be modified to a cycle of length exactly ℓ , using the property of P .

5.1 Lemmas

We shall need the following lemma, that builds on Lemma 3.1, and yields a path with many chords that can be extended to a cycle of length at most n/α .

Lemma 5.1. *Let $0 < \delta \ll 1$, and let α, n be positive integers satisfying: α and n/α are large; $n \leq 4\alpha^2$; and $n \geq \alpha$. Suppose that G is a graph on n vertices, with $\alpha(G) = \alpha$ and $\kappa(G) > \alpha$. Then there are paths P_0 and P_1 , which have the same ends and vertex-disjoint interiors, such that: $\ell(P_0) \leq 7\delta n/\alpha$; $\ell(P_1) \geq \delta n/\alpha$; $\ell(P_0) + \ell(P_1) \leq n/\alpha$; and P_0 has at least $\delta n/\alpha$ non-intersecting chords of length 2 or 3, with at least one of length 2.*

We will also use the following lemma, about extending a given path by a relatively small amount.

Lemma 5.2. *Let α, r, ℓ, n be positive integers satisfying: α is large; $\max\{4\sqrt{\ell}, 32\sqrt{\alpha}\} < r \leq \min\{2\ell, n/\alpha, \alpha\}$; and $\ell \leq \frac{n}{2}$. Suppose that G is a graph on n vertices, satisfying $\alpha(G) \leq \alpha$ and $\kappa(G) \geq \alpha/2$, and that P is a path of length ℓ in G . Then there is a path P' with the same ends as P that satisfies $|P| < |P'| \leq |P| + r$.*

We first show how to prove Lemma 2.2 using these lemmas, and then we prove both lemmas.

5.2 Proof of middle range lemma

Using the above lemmas, the proof of Lemma 2.2 is straightforward, albeit somewhat technical.

Proof of Lemma 2.2. Let η be a constant satisfying $\delta \ll \eta \ll 1$. Apply Lemma 5.1 with η to find paths P_0 and P_1 with the same ends, denoted x, y , and disjoint interiors, such that: $\ell(P_0) \leq 7\eta n/\alpha$; $\ell(P_1) \geq \eta n/\alpha$; $\ell(P_0) + \ell(P_1) \leq n/\alpha$; and P_0 has $\eta n/\alpha$ non-intersecting chords of length 2 or 3, with at least one of length 2. Consider the graph G_1 , obtained from G by removing the interior vertices of P_0 . Then $\alpha(G_1) \leq \alpha$, $|G_1| \geq n/2$, and, since $\ell(P_0) \leq 7\eta n/\alpha \leq \alpha/2$ (using $\alpha \geq \sqrt{n}/2$), we also have $\kappa(G) \geq \alpha/2$. Write $\ell' = \ell - \ell(P_0)$ and $r = \eta n/2\alpha$. Let P_2 be a longest xy -path in G_1 of length at most $\ell' + r$.

Claim 5.3. $\ell(P_2) \geq \ell'$.

Proof. Suppose not. Then apply Lemma 5.2 with $\alpha, r, \ell_{5.2} = \ell(P_2)$ and $n_{5.2} = |G_1|$. To see that the lemma is applicable, notice that, by assumption, $\ell(P_2) \leq \ell' \leq \ell \leq n/4 \leq |G_1|/2$, and $\kappa(G_1) \geq \alpha/2$. Moreover, we now verify that $\max\{4\sqrt{\ell''}, 32\sqrt{\alpha}\} \leq r \leq \min\{2\ell'', |G_1|/\alpha, \alpha\}$, where $\ell'' = \ell(P_2)$:

- $4\sqrt{\ell''} \leq 4\sqrt{\ell} \leq 4\sqrt{\delta n}/\alpha \leq \eta n/2\alpha = r$, using the assumption $\ell'' \leq \ell'$ and that $\ell' \leq \ell$.
- $\frac{32\sqrt{\alpha}}{r} = \frac{64\alpha^{3/2}}{\eta n} \leq \frac{64\delta^{3/2}}{\eta} \leq 1$, using $\alpha \leq \delta n^{2/3}$, and showing $32\sqrt{\alpha} \leq r$.
- $2\ell'' \geq 2\ell(P_1) \geq 2\eta n/\alpha \geq r$, by maximality of P_2 .
- $\frac{|G_1|}{\alpha} \geq \frac{n}{2\alpha} \geq \frac{\eta n}{2\alpha} = r$.
- $\frac{\alpha}{r} = \frac{2\alpha^2}{\eta n} \geq \frac{1}{2\eta} \geq 1$, using $\alpha \geq \sqrt{n}/2$, and showing $\alpha \geq r$.

This shows that the lemma is indeed applicable. It implies the existence of an xy -path P'_2 with $\ell(P_2) < \ell(P'_2) \leq \ell(P_2) + r \leq \ell' + r$, contradicting the maximality of P_2 . \square

Note that $\ell - \ell(P_2) \leq \ell - \ell' = \ell(P_0)$ (by the claim) and $\ell - \ell(P_2) \geq \ell - \ell' - r = \ell(P_0) - \eta n/2\alpha$ (by choice of P_2). Recalling that P_0 has $\eta n/\alpha$ non-intersecting chords of length 2 or 3, with at least one of length 2, by Proposition 3.2 there is an xy -path P'_0 of length $\ell - \ell(P_2)$ with $V(P'_0) \subseteq V(P_0)$. The concatenation of P'_0 and P_2 is a cycle of length exactly ℓ . \square

5.3 Finding a short cycle with many short chords

We now prove Lemma 5.1. We start by taking a path P_{init} which is relatively short yet not too short, and has many chords of length 2 or 3, with at least one of length 2; denote its ends by x, z . Now we consider the graph G' obtained by removing the interior of P_{init} , and observe that it has connectivity at least a little below α . If the distance between x and z in G' is sufficiently smaller than n/α , we are done. If not, then due to space considerations, for almost all $i \leq \alpha$, there are roughly n/α vertices at distance exactly i from x in G' . In particular, there is a relatively small i such that the number of vertices at distance j from x in G' is indeed roughly n/α , for $j \in [i, i + 6]$. We then focus on the vertices at distance between i and $i + 6$ from x in G' , and repeat arguments similar to the ones used in Lemma 3.1, to find the desired paths.

Proof of Lemma 5.1. By Lemma 3.1, there is a path Q_1 of length at most $\frac{3\delta n}{\alpha}$ with at least $\frac{\delta n}{\alpha}$ non-intersecting chords of length 2 or 3, at least one of which having length 2. Denote the ends of Q_1 by x, y . By the connectivity assumption, there is a path Q_2 of length $\frac{\delta n}{\alpha}$, one of whose ends is y , and which otherwise does not intersect Q_1 . Let z be the other end of Q_2 . Write $P_{\text{init}} = xQ_1yQ_2z$; so $\ell(P_{\text{init}}) \leq \frac{4\delta n}{\alpha} \leq 16\delta\alpha$, using $n \leq 4\alpha^2$.

Consider the graph G' , obtained from G by removing the interior vertices of P_{init} . Write

$$\kappa = (1 - 16\delta)\alpha, \quad \ell = (1 - 4\delta) \cdot \frac{n}{\alpha},$$

and notice that $\kappa(G') \geq \alpha - |P_{\text{init}}| \geq \kappa$. If G' has an xz -path Q_3 of length at most ℓ , we are done (take $P_0 = Q_1$ and $P_1 = yQ_2zQ_3x$), so suppose it does not.

Let U_i be the set of vertices at distance exactly i from x in G' . Notice that U_i is non-empty for $i \leq \ell$. By connectivity of G' , we have $|U_i| \geq \kappa$ for $i \in [\ell - 1]$. Moreover, there is a matching of size at least κ between U_i and U_{i+1} , for $i \in [\ell - 2]$. We claim that there exists i with $3 \leq i \leq \frac{\eta n}{\alpha}$ such that the sets $U_i, U_{i+1}, \dots, U_{i+6}$ all have size at most $(1 + \eta)\alpha$. Indeed, otherwise, we get

$$\begin{aligned} \sum_{i \in [\ell]} |U_i| &\geq \kappa \left(\ell - \frac{\eta n}{\alpha} \right) + \left(\frac{\eta n}{7\alpha} - 1 \right) \cdot (6\kappa + (1 + \eta)\alpha) \\ &\geq \kappa \ell - \frac{\kappa \eta n}{7\alpha} + \frac{\eta n}{7} \cdot (1 + \eta) - 7(1 + \eta)\alpha \\ &= \kappa \ell + \frac{\eta n}{7} \cdot (1 + \eta - (1 - 16\delta)) - 7(1 + \eta)\alpha \\ &\geq (1 - 16\delta)(1 - 4\delta)n + \frac{\eta^2 n}{7} - 8\alpha \geq \left(1 - 20\delta + \frac{\eta^2}{7} - \frac{8}{n/\alpha} \right) n > n, \end{aligned}$$

a contradiction. For the first estimate we lower bounded $|U_i|$ by κ for $i > \eta n/\alpha$, and divided the range $[3, \eta n/\alpha]$ into at least $\eta n/7\alpha - 1$ intervals of length 7, and then lower bounded the total length of $|U_i|$ over each of these intervals by $6\kappa + (1 + \eta)\alpha$, which was possible due to the assumption that in each such interval there is a set U_i of size at least $(1 + \eta)\alpha$. For the second inequality we used $\kappa \leq \alpha$, and for the last estimate we used that n/α is large.

Fix an i as above. Let M_j be a matching of size κ between U_{i+j-1} and U_{i+j} , for each $j \in [6]$, and denote the union of these matchings by F . Let U'_{i+j} be the set of vertices in U_{i+j} that are in a path of length 6 in this union. We claim that $|U'_{i+j}| \geq (1 - 13\eta)\alpha$. To see this, consider the set of vertices U''_{i+k} in U_{i+k} which are in paths of length $k - 1$ in F that start in U_i . Then $|U''_i| \geq \kappa \geq (1 - \eta)\alpha$, and $|U''_{i+j}| \geq |U''_{i+j-1}| - (|U_{i+j}| - \kappa) \geq |U''_{i+j-1}| - 2\eta\alpha \geq (1 - (2j + 1)\eta)\alpha$. Each vertex in U''_{i+6} is in a path of length 6. Thus $|U'_{i+j}| \geq |U''_{i+6}| \geq (1 - 13\eta)\alpha$, for $j \in [0, 6]$.

Rename $W_j = U'_{i+j}$ for $j \in [0, 6]$, and write $W = W_0 \cup \dots \cup W_6$. Notice that every vertex in $W_1 \cup \dots \cup W_5$ has at least $\kappa - 14\eta\alpha \geq \alpha/2$ neighbours in W . For a vertex $w \in W_j$ let w^+ be the F -neighbour of w in W_{j+1} , for $j \in [0, 5]$, and let w^- be the F -neighbour of w in W_{j-1} , for $j \in [6]$. Moreover, let $P(w)$ be the unique path through w from W_0 to W_6 in F . Write $\ell' = \frac{\delta n}{\alpha}$.

Claim 5.4. *There is a sequence of paths $P_0, \dots, P_{2\ell'}$ with the following properties.*

- (a) *The paths P_i are pairwise vertex-disjoint, except that the start point of P_i is the end point of P_{i-1} , for $i \in [2\ell']$.*
- (b) $|P_i| \leq 8$.
- (c) *P_i has a chord of length 2 or 3 for $i \in [0, 2\ell']$ (2 if $i = 0$).*
- (d) *The end point of P_i is in $W_1 \cup \dots \cup W_5$.*
- (e) *$V(P_i) \subseteq W$ for $i \in [2\ell']$. If $V(P_0)$ is not fully contained in W , then it starts with a vertex in P_{init} , such that the subpath of P_{init} between x and P_0 's start point does not contain other vertices of P_0 .*

Proof. First, define P_0 as follows. Let $u \in W_3$. By assumption on G , the vertex u lies in a triangle (uvw) . If one of v and w is in P_{init} , then without loss of generality v is in P_{init} and is closer to x than w in P_{init} (if w is also in P_{init}). Define $P_0 = vwu$, and notice that P_0 satisfies the above requirements.

Now suppose that P_0, \dots, P_{i-1} are defined and satisfy the above properties, where $i \in [\ell']$. Denote the end point of P_{i-1} by u ; then $u \in W_1 \cup \dots \cup W_5$ by (e). Let W' be the set of vertices in W which do not lie in any path $P(v)$, with $v \in V(P_0) \cup \dots \cup V(P_i)$. Then $|W'| \geq |W| - 8\ell' \geq |W| - \frac{\alpha}{4}$. Let v be a neighbour of u in W' , let $P'(v)$ be the subpath of $P(v)$ that starts at v and ends in W_3 , and let w be the end of $P'(v)$. As in the proof of Lemma 3.1, there exist $w_1, w_2, w_3 \in W' - V(P(v))$ such that one of the following holds: (ww_1w_2) is a triangle; $(w_1w_2w_3)$ is a triangle and ww_1 is an edge; or $(ww_1w_2w_3)$ is a 4-cycle. Let Q be ww_1w_2 in the first case, and $ww_1w_2w_3$ in the other two cases, and define $P_{i+1} = uvP'(v)wQ$.

It is easy to check that P_i satisfies the requirements. This completes the proof that paths $P_0, \dots, P_{\ell'}$ as above exist. \square

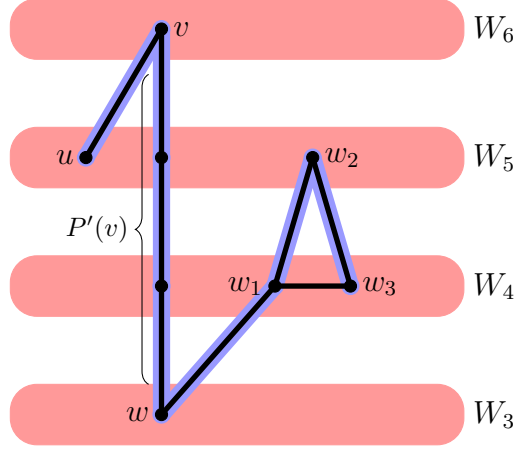


Figure 6: An illustration of how P_i may look like

Define $P = P_0 \dots P_{2\ell'}$, and denote the start point of P by u_0 and the end point by u_1 .

We define paths Q_0 and Q_1 between x and u_0 and u_1 . Let $i \in \{0, 1\}$. If $u_i \in W$, let v_i be a neighbour of u_i such that $P(v_i)$ does not intersect $P(v)$ for any v in P (it is easy to see that such v_i exists), and let Q_i be the concatenation of a shortest path between x and v_i that does not intersect P and the edge $v_i u_i$. Then $\ell(Q_i) \leq \frac{2\eta n}{\alpha}$. If $u_i \notin W$, then $i = 0$ and $u_i \in V(P_{\text{init}})$, and we take Q_0 to be the subpath of P_{init} between x and u_0 . In this case we have $\ell(Q_0) \leq \ell(P_{\text{init}}) \leq \frac{4\delta n}{\alpha}$.

Thinking of Q_0 as a path from x to u_0 , let x' be the last vertex in Q_0 which is also in Q_1 (notice that x is in both Q_0 and Q_1 , so this is well defined). Let Q'_0 be the subpath of Q_0 from x' to u_0 , and let Q'_1 be the subpath of Q_1 from u_1 to x' . The paths $P_0 \dots P_{\ell'-1}$ (for P_0) and $P_{\ell'} \dots P_{2\ell'} Q_1 Q_0$ (for P_1) satisfy the requirements of the lemma. Indeed, the former has length at most $7\delta n/\alpha$ and at least $\delta n/\alpha$ non-intersecting chords of length 2 or 3, with at least one of length 2, the latter has length at least $\delta n/\alpha$, and the total length is at most $(14\delta + 4\eta)n/\alpha \leq n/\alpha$. \square

5.4 Extending middle range cycles

In this section we prove Lemma 5.2. Here the task is to extend a given path P by a small but positive amount. To do so, we use results about cycle-complete Ramsey numbers to find a cycle C of specific length which is disjoint of P . We then use connectivity to find many pairwise vertex-disjoint paths between P and C , one for each vertex of C . If many of the paths are short, we can pick two that are close in P , and extend P by going along these paths and along the longer subpath of C that connects the ends of these paths in P (see Figure 7a). Otherwise, we use the independence number to find an edge between two of the paths to get an appropriate extension (see Figure 7b). The proof here is reminiscent of the proof of Lemma 2.10 in [14], especially in the latter case, but the idea to use a cycle outside of P is new and crucial for covering the full middle range.

Proof of Lemma 5.2. By Theorem 3.4, the bound $r \geq 32\sqrt{\alpha}$, and the assumption that α is large, we have the following bound on the cycle-complete Ramsey number: $r(C_{r/2}, K_{\alpha+1}) \leq (r/2) \cdot \alpha \leq n/2 \leq n - \ell$. Since $\alpha(G) \leq \alpha$, it follows that there is a cycle $C = (v_1 \dots v_{r/2})$ of length $r/2$ which is

vertex-disjoint of P . Since $\kappa(G) \geq \alpha/2 \geq r/2 = |C|$ and $|P| = \ell + 1 \geq r/2$, there are $r/2$ pairwise vertex-disjoint paths from C to P . Given such a collection of paths, denote the path starting with v_i by Q_i , for $i \in [r/2]$, and let u_i be the end of Q_i in P . By assuming that the Q_i 's are as short as possible, the paths Q_i are induced. We consider two cases: at least $r/4$ paths P_i have length at most $r/4$; and at least $r/4$ paths have length at least $r/4$.

Suppose that the former holds, and let I be the set of indices $i \in [r/2]$ such that $\ell(Q_i) \leq r/4$. Since $|I| \geq r/4$, there are distinct $i, j \in I$ such that the distance between u_i and u_j in P is at most $\frac{\ell}{r/4} \leq r/4$ (using $r \geq 4\sqrt{\ell}$). Let P_1 and P_2 be the two subpaths of P obtained by removing the segment between u_i and u_j , such that $u_i \in V(P_1)$ and $u_j \in V(P_2)$. Let P_3 be the longer subpath of C with ends v_i and v_j ; so $\ell(P_3) \geq r/4$. Define P' as follows (see Figure 7a).

$$P' = P_1 u_i Q_i v_i P_3 v_j Q_j u_j P_2.$$

We claim that P' satisfies the requirements. Indeed, notice that P' has the same ends as P . Moreover, $\ell(P') \geq \ell(P) - r/4 + \ell(Q_i) + \ell(Q_j) + \ell(P_3) \geq \ell(P) + 2$, as at most $r/4$ edges of P are skipped, the paths Q_i and Q_j each have length at least 1, and P_3 has length at least $r/4$. Also, $\ell(P') \leq \ell(P) + \ell(Q_i) + \ell(Q_j) + \ell(P_3) \leq \ell(P) + r$, using $\ell(Q_i), \ell(Q_j) \leq r/4$ and $\ell(P_3) \leq r/2$.

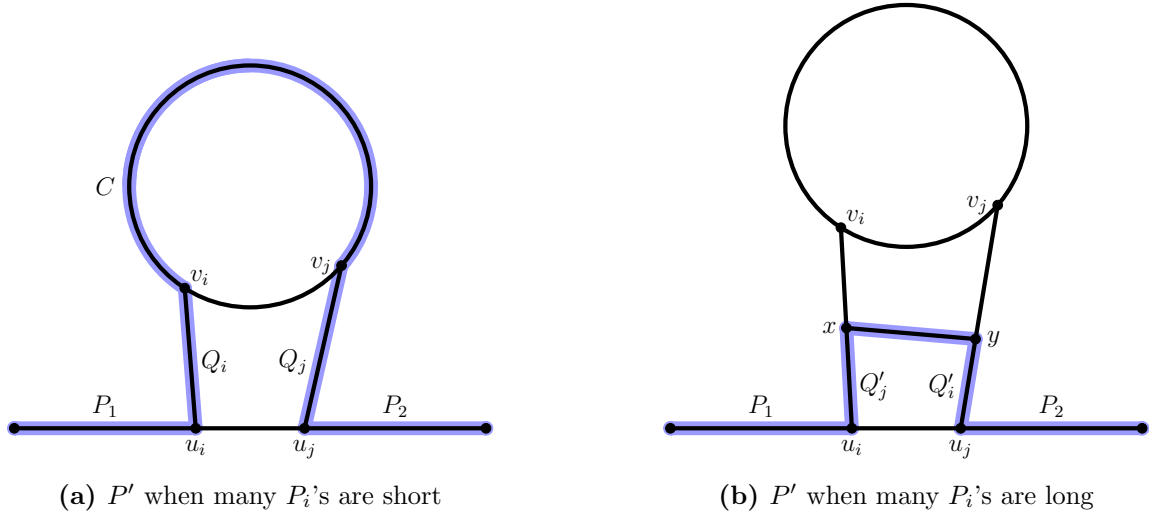


Figure 7: the path P'

Now suppose that the latter case holds, namely at least $r/4$ of the paths P_i have length at least $r/4$; let I be the set of indices i such that $\ell(P_i) \geq r/4$. We claim that there is a subpath of P of length at most $r/8$ that contains at least $32\alpha/r$ vertices u_i , with $i \in I$. Indeed, otherwise

$$\ell \leq \left(\frac{\ell}{r/8} + 1 \right) \cdot \frac{32\alpha}{r} < \frac{2^9\alpha}{r^2} \cdot \ell < \ell,$$

a contradiction (using $r \geq 32\sqrt{\alpha}$). Let J be a subset of I of size at least $32\alpha/r$ such that the vertices u_j with $j \in J$ are on a subpath of P of length at most $r/8$. For each $j \in J$, let U_j be the set of vertices in P_j whose distance from u_j in P_j is *even* and between $r/8$ and $r/4$. Then U_j is an independent set (as P_j is an induced path) of size at least $r/16$. It follows that the union $\bigcup_{j \in J} U_j$ has size at least

$(r/16) \cdot (32\alpha/r) > \alpha$. As $\alpha(G) \leq \alpha$, there is an edge xy in this union; say $x \in U_i$ and $y \in U_j$ (so $i \neq j$). Let Q'_i be the subpath of Q_j from u_j to x , let Q'_j be the subpath of Q_j from y to u_j , and let P_1 and P_2 be the two subpaths of P obtained by removing the segment between u_i and u_j , such that $u_i \in V(P_1)$. Define a path P' as follows (see Figure 7b).

$$P' = P_1 u_i Q'_i x y Q'_j u_j P_2.$$

Again, we claim that P' satisfies the requirements. Indeed, it has the same ends as P . Moreover, $\ell(P') \geq \ell(P) - r/8 + \ell(Q'_i) + \ell(Q'_j) + 1 > \ell(P)$ (using $\ell(Q'_i), \ell(Q'_j) \geq r/8$), and $\ell(P') \leq \ell(P) + \ell(Q'_i) + \ell(Q'_j) + 1 \leq \ell(P) + r$ (using $\ell(Q'_i), \ell(Q'_j) \leq r/4$). \square

6 Lower range

In this section we prove Lemma 2.3, restated here, about short cycles.

Lemma 2.3 (Lower range). *Let $0 < \delta \ll 1$ and let $n, \alpha, \ell \geq 1$ be integers, satisfying: α is sufficiently large, and $3 \leq \ell \leq \max\{n/\alpha, \delta\alpha\}$. Suppose that G is an n -vertex graph with $\delta(G) > \alpha(G) = \alpha$. Then G has a cycle of length ℓ .*

The general outline is very similar to that of the proof of the lower range in [14], but we need to work harder for each step. We find very short cycles (of length up to 7) via ad-hoc arguments; for comparison, in [14] the authors needed to find cycles of length up to 5, and their proofs do not quite carry to our setting.

To find longer cycles in this range, our proof splits into two cases. When $n/\alpha \geq \delta\alpha$, we use results about cycle-complete Ramsey numbers (Theorem 3.3 and Theorem 3.4), exactly as in [14]. Otherwise, we use a result about the Turán number of even cycles due to Bondy and Simonovits [7] (see Theorem 6.2). This works directly for even lengths, but for odd lengths we need a correction. Lemma 6.3 provides such a correction: it finds a large subgraph $H \subseteq G$ such that every edge e in H can be extended to a cycle of length 3 or 5 in G , whose vertices outside of e are not in H . This is a variant of Lemma 2.7 in [14], where the stronger condition on the connectivity allowed the authors to find such H where every edge e is in a triangle in G whose third vertex is outside of H .

6.1 Lemmas

For technical reasons, we will find very short cycles (of length at most 7) separately.

Lemma 6.1. *Let G be a graph with $\delta(G) > \alpha(G) \gg 1$. Then G contains a cycle of length ℓ , for every $\ell \in \{3, \dots, 7\}$.*

For longer cycles we will consider two cases, depending on whether or not n/α is larger than $\delta\alpha$. If yes, then the proof follows directly from the results we mentioned earlier regarding cycle-complete Ramsey numbers (Theorems 3.3 and 3.4). If not, then we use the following result about the Turán number of even cycles.

Theorem 6.2 (Bondy–Simonovits [7]). *Let $\ell \geq 1$ be an integer, and suppose that G is an n -vertex graph with $e(G) \geq \max\{20\ell n^{1+1/\ell}, 200n\ell\}$. Then G contains a cycle of length 2ℓ .*

The following lemma allows us to find odd cycles in the latter case, by applying Theorem 6.2 to a subgraph H of G which has the property that every edge e is in a short odd cycle in G whose vertices which are not in e are not in H .

Lemma 6.3. *Let $0 < \delta \ll 1$ and let $\alpha, n \geq 1$ be integers. Suppose that G is an n -vertex graph with $\delta(G) > \alpha(G) = \alpha$. Then there is a subgraph H of G with at least $\delta n \alpha$ edges, such that each edge e in H is contained in a cycle C of length 3 or 5 satisfying $V(C) \cap V(H) = V(e)$.*

6.2 Proof of lower range lemma

Proof of Lemma 2.3. Let η be a constant satisfying $\delta \ll \eta \ll \alpha$. Write $m = \max\{n/\alpha, \delta\alpha\}$, and let ℓ be an integer satisfying $3 \leq \ell \leq m$. If $\ell \leq 7$ then there is a cycle of length ℓ , by Lemma 6.1, so we may assume $\ell \geq 8$. We consider two cases, according to the value of m .

Suppose first that $m = n/\alpha$. Since G has no independent set of size $\alpha + 1$ and α is large, there is a cycle of length exactly ℓ , using aforementioned results about cycle-complete Ramsey numbers, namely Theorem 3.3 if $\ell \leq \log \alpha$ and Theorem 3.4 otherwise.

Suppose now that $m = \delta\alpha$. Apply Lemma 6.3 to find a subgraph H of G with at least $\eta n \alpha$ edges such that for every edge $e \in E(H)$ there is a cycle C_e of length 3 or 5 that contains e and whose vertices which are not in e are not in H . Let H_i be the subgraph of H with edges e such that $|C_e| = i$, for $i \in \{3, 5\}$. Let $i \in \{3, 5\}$ be such that $e(H_i) \geq \eta n \alpha / 2$. Write

$$k = \begin{cases} \frac{\ell-i+2}{2} & \ell \text{ is odd} \\ \frac{\ell}{2} & \ell \text{ is even.} \end{cases}$$

Claim 6.4. $e(H_i) \geq \max\{20kn^{1+1/k}, 200nk\}$.

Proof. First note that $200nk \leq 200n\ell \leq 200\delta n \alpha \leq \eta n \alpha / 2 \leq e(H_i)$.

Next, note that, as $\ell \geq 8$, we have $k \geq 3$. Moreover, since $m = \delta\alpha$, we have $\delta\alpha \geq n/\alpha$, so $\alpha \geq \delta^{-1/2} \sqrt{n}$. This implies $e(H_i) \geq \eta \delta^{-1/2} n^{3/2} / 2 \geq n^{3/2}$. Now, if $k \geq \log n$ then $20kn^{1+1/k} \leq 20 \cdot e \cdot kn \leq e(H_i)$. If, instead, $3 \leq k \leq \log n$, then $20kn^{1+1/k} \leq 20 \log n \cdot n^{4/3} \leq n^{3/2} \leq e(H_i)$, using that α and thus n are large. \square

By the claim and Theorem 6.2, the graph H_i has a cycle of length exactly $2k$. If ℓ is even, then this is a cycle of length ℓ in G . If ℓ is odd, then let C be a cycle of length $2k = \ell - (i - 2)$ in H_i . Let C' be the cycle obtained by replacing one edge e in C by the subpath of C_e obtained by removing e . Notice that this is indeed a cycle, as the vertices of C_e which are not in e are not in H_i , and thus not in C . Also, $|C'| = |C| - 1 + (i - 1) = \ell$. \square

6.3 Finding short cycles

We now prove Lemma 6.1, about finding cycles of length between 3 and 7.

Proof of Lemma 6.1. Write $\alpha = \alpha(G)$. For each vertex u , let $I(u)$ be a largest independent set in the neighbourhood $N(u)$.

$\ell = 3$. Since $|N(u)| > \alpha$ for every vertex u , there is an edge in $N(u)$, showing that G contains a triangle.

$\ell = 4$. Notice that if, for some vertex u , its neighbourhood $N(u)$ contains a path $v_1v_2v_3$ of length 2, then $(uv_1v_2v_3)$ is a cycle of length 4. So suppose that $N(u)$ does not contain a path of length 2, for every vertex u . In particular, each neighbourhood $N(u)$ induces a matching, and thus $|I(u)| \geq \alpha/2$. Consider a triangle (uvw) . If $|I(u) \cap I(v)| \geq 2$, then take a vertex $x \in (I(u) \cap I(v)) - \{w\}$, and note that $(uxvw)$ is a 4-cycle (see Figure 8a). So suppose that $|I(u) \cap I(v)|, |I(u) \cap I(w)|, |I(v) \cap I(w)| \leq 1$. Thus $|I(u) \cup I(v) \cup I(w)| \geq 3\alpha/2 - 3 > \alpha$, showing that $I(u) \cup I(v) \cup I(w)$ contains an edge xy . By independence of $I(u), I(v), I(w)$, and by symmetry, we may assume $x \in I(u)$ and $y \in I(v)$. Thus $(uxyv)$ is a 4-cycle (see Figure 8b).



Figure 8: 4-cycles

$\ell = 5$. To see that G contains a 5-cycle, we may assume that $N(u)$ does not contain a path of length 3, for every vertex u . It follows that $|I(u)| \geq \alpha/3$. Suppose that some neighbourhood $N(u)$ contains a matching $\{v_1w_1, \dots, v_4w_4\}$ of size 4. If $|I(v_i) \cap I(v_j)| \geq 10$ for some distinct $i, j \in [4]$, pick $x \in (I(v_i) \cap I(v_j)) - \{v_1, w_1, \dots, v_4, w_4, u\}$, then $(uv_ixv_jw_j)$ is a 5-cycle (see Figure 9a). Otherwise, the set $I(v_1) \cup \dots \cup I(v_4)$ has size at least $4\alpha/3 - 6 \cdot 9 > \alpha$, and so it has an edge xy , say $x \in I(v_i)$ and $y \in I(v_j)$. Notice that $i \neq j$, by independence of the sets $I(v_1), \dots, I(v_4)$. Then (uv_ixyv_j) is a 5-cycle (see Figure 9b). Now suppose that no neighbourhood $N(u)$ contains a matching of size 4, showing that $|I(u)| \geq \alpha - 6$ for every vertex u . Let $(v_1v_2v_3)$ be a triangle. If there is an edge xy with $x \in I(v_i)$, $y \in I(v_j)$, and $x, y \notin \{v_1, v_2, v_3\}$, then, without loss of generality $i = 1$ and $j = 2$, and so $(v_1xyv_2v_3)$ is a 5-cycle (see Figure 9c). So suppose no such edge exists. This shows that $|I(v_i) \cup I(v_j)| \leq \alpha$ and thus $|I(v_i) \cap I(v_j)| \geq \alpha - 12$ for all $i, j \in [3]$. In particular, there are distinct vertices $x \in I(v_1) \cap I(v_2)$ and $y \in I(v_2) \cap I(v_3)$, such that $x, y \notin \{v_1, v_2, v_3\}$. Then $(v_1xv_2yv_3)$ is a 5-cycle (see Figure 9d).

$\ell = 6$. To find a 6-cycle, we may assume as above that $|I(u)| \geq \alpha/4$ for every vertex u . Suppose that some neighbourhood $N(u)$ contains a matching $\{v_1w_1, \dots, v_5w_5\}$ of size 5. Then, similarly to the previous paragraph, there are distinct $i, j \in [5]$ and either a vertex $x \in (I(v_i) \cap I(v_j)) - \{v_1, w_1, \dots, v_5, w_5, u\}$,

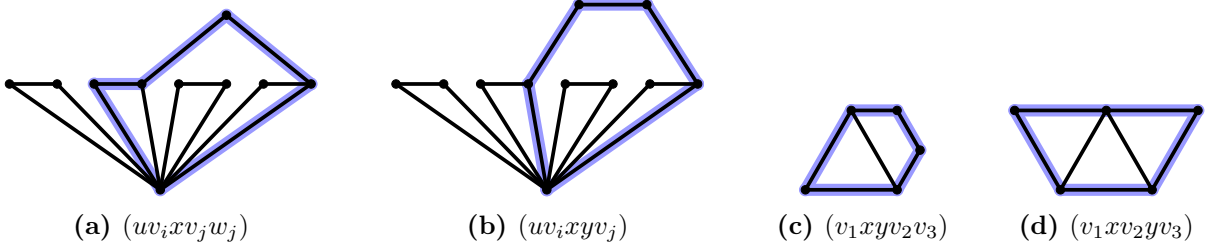


Figure 9: 5-cycles

or two vertices $x \in I(v_i) - \{v_1, w_1, \dots, v_5, w_5, u\}$ and $y \in I(v_j) - \{v_1, w_1, \dots, v_5, w_5\}$ such that xy is an edge. This yields a 6-cycle: $(uw_i v_i x v_j w_j)$ in the first case, and $(uv_i x y v_j w_j)$ in the second case (see Figures 10a and 10b). So we now may assume $|I(u)| \geq \alpha - 8$ for every vertex u . Consider a triangle $(v_1 v_2 v_3)$. If $|I(v_i) \cap I(v_j)| \geq 6$ for every distinct $i, j \in [3]$, we may find a 6-cycle of the form $(v_1 x v_2 y v_3 z)$ (see Figure 10c). Otherwise, say $|I(v_1) \cap I(v_2)| \leq 6$, and so $|I(v_1) \cup I(v_2)| \geq 2\alpha - 22$. In particular, there is a matching $\{x_1 y_1, x_2 y_2\}$ with $x_1, x_2 \in I(v_1)$ and $y_1, y_2 \in I(v_2)$, yielding the 6-cycle $(v_1 x_1 y_1 v_2 y_2 x_2)$ (see Figure 10d).

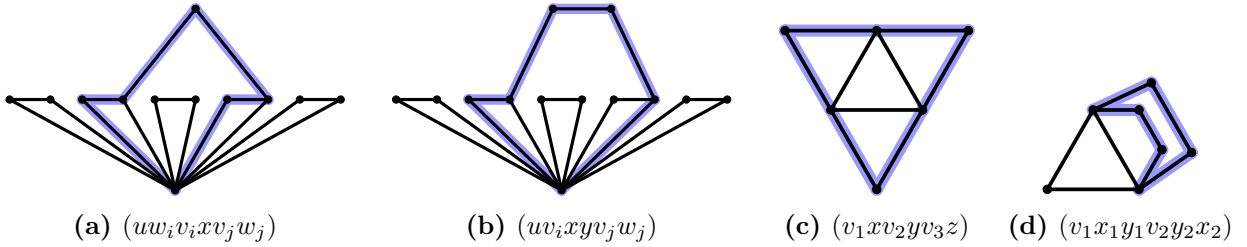


Figure 10: 6-cycles

$\ell = 7$. Finally, we need to prove the existence of a 7-cycle. As usual, we assume $|I(u)| \geq \alpha/5$ for every vertex u . Suppose $N(u)$ contains a matching $\{v_1 w_1, \dots, v_6 w_6\}$. If there is an edge xy with $x \in I(v_i)$ and $y \in I(v_j)$ (and $x, y \notin \{v_1, w_1, \dots, v_6, w_6, u\}$), then we get a 7-cycle $(uw_i v_i x y v_j w_j)$ (see Figure 11a). So suppose no such edge exists, showing that $|I(v_1) \cup \dots \cup I(v_6)| \leq \alpha$. This implies that, for some distinct $i, j \in [6]$, we have $|I(v_i) \cap I(v_j)| \geq 10$. Let $x_1, \dots, x_6 \in (I(v_i) \cap I(v_j)) - \{v_i, v_j, w_j, u\}$. As usual, there are distinct $s, t \in [6]$ such that either there exists $y \in (I(x_s) \cap I(x_t)) - \{v_i, v_j, w_j, u\}$, or there is an edge yz with $y \in I(x_s) - \{v_i, v_j, w_j, u\}$ and $z \in I(x_t) - \{v_i, v_j, w_j, u\}$. Either way, there is a 7-cycle: $(uv_i x_i y x_j v_j w_j)$ in the first case, and $(uv_i x_i y z x_j v_j)$ in the second (see Figures 11b and 11c). We may now assume that $|I(u)| \geq \alpha - 10$ for every vertex u . This implies that for every two vertices a, b and any set S of size at most 7 which does not contain a or b , there is a path from a to b that avoids S and has length 2 or 3. Consider a triangle $(v_1 v_2 v_3)$, and for $1 \leq i < j \leq 3$, let $P_{i,j}$ be a path of length 2 or 3 with ends v_i and v_j , such that distinct paths have disjoint interiors. If at least two of these paths have length 3, say $\ell(P_{1,2}), \ell(P_{2,3}) = 3$, then $(v_1 P_{1,2} v_2 P_{2,3} v_3)$ is a 7-cycle (see Figure 11d). If exactly one path has length 3, say $\ell(P_{1,2}) = 3$, then $(v_1 P_{1,2} v_2 P_{2,3} v_3 P_{1,3})$ is a 7-cycle (see Figure 11e). We may thus assume that all paths have length 2, i.e. there are distinct vertices u_1, u_2, u_3 such that $(v_1 u_1 v_2 u_2 v_3 u_3)$ is a 6-cycle. Let P be a path of length 2 or 3, with ends v_1, u_1 , which avoids $\{v_2, u_2, v_3, u_3\}$. If P has

length 2 then $(v_1 P u_1 v_2 u_2 v_3 u_3)$ is a 7-cycle, and otherwise $(v_1 P u_1 v_2 u_2 v_3)$ is a 7-cycle (see Figures 11f and 11g). \square

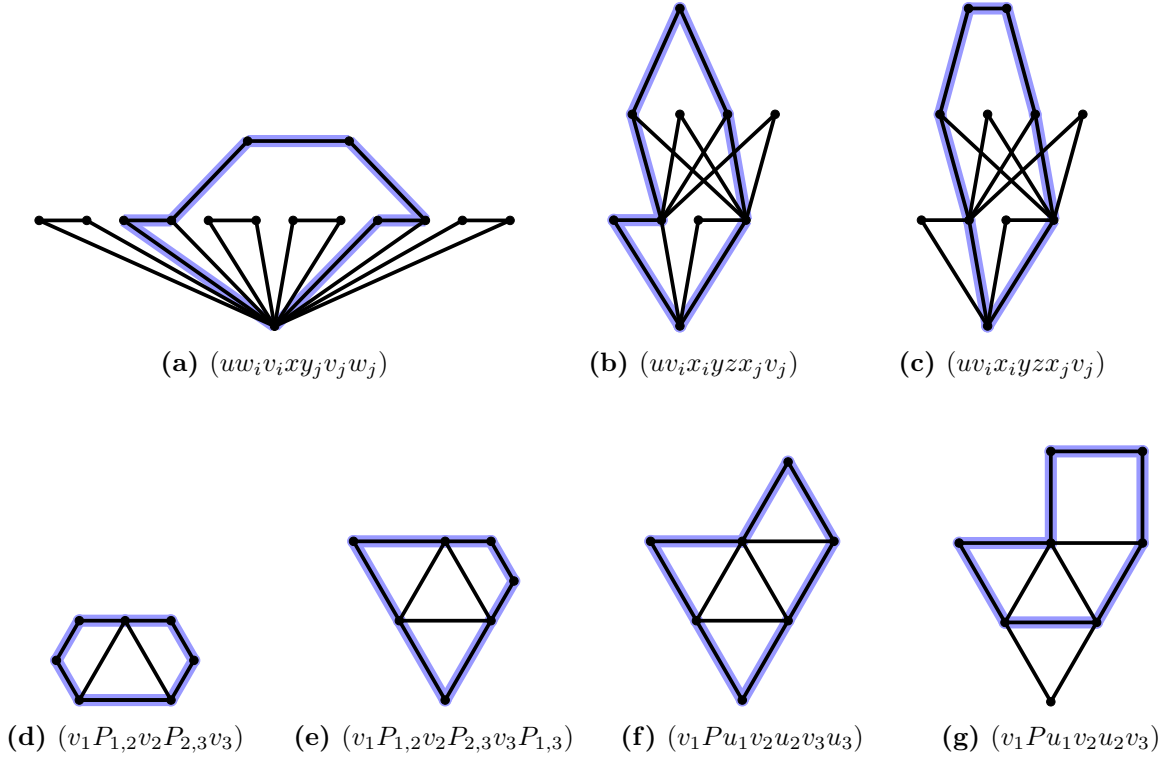


Figure 11: 7-cycles

6.4 Finding many edges in short odd cycles

We now prove Lemma 6.3. The idea is to find $\Omega(n\alpha)$ edges in G that are contained in either a triangle or a 5-cycle. Once that is done, a simple probabilistic argument can be used to find a subgraph H with the desired properties.

Proof of Lemma 6.3. Let η be a constant satisfying $\delta \ll \eta \ll 1$. For each vertex u let $M(u)$ be a maximum matching in $N(u)$, and let $I(u) := N(u) - M(u)$ (so $I(u)$ is independent). We define sets U_1, \dots, U_4 as follows.

- Let U_1 be the set of vertices u satisfying $|M(u)| \geq \alpha/12$.
- Let U_2 be the set of vertices $u \notin U_1$ for which there exists a triangle (uvw) , where $v \notin U_1$.
- Let U_3 be the set of vertices $u \notin U_1 \cup U_2$ for which there exist three vertices $v, w, x \neq u$ such that (vwx) is a triangle, $v \notin U_1$, and uw is an edge.
- Let $U_4 = V(G) - (U_1 \cup U_2 \cup U_3)$.

We show that $|U_4| \leq n/3$. Indeed, for each $u \in U_4$ let T_u be a triangle that contains u (such a triangle exists by $\delta(G) > \alpha$). We claim that the triangles T_u , with $u \in U_4$, are pairwise vertex-disjoint. To see this, consider distinct $u_1, u_2 \in U_4$, and write $T_{u_1} = (u_1 v_1 w_1)$ and $T_{u_2} = (u_2 v_2 w_2)$. Notice that $v_1, w_1, v_2, w_2 \in U_1$, because $u_1, u_2 \notin U_2$. Thus, if T_{u_1} and T_{u_2} share a vertex, we may assume $v_1 = v_2$. This is a contradiction to $u \notin U_4$, as $(u_2 v_2 w_2)$ is a triangle, $u_2 \notin U_1$, and $u_1 v_2$ is an edge. This shows that the triangles T_u , with $u \in U_4$, are pairwise vertex-disjoint, which implies the desired inequality $|U_4| \leq n/3$.

Let \mathcal{F} be a family of ordered triples and quintuples, defined as follows. It will be useful to note that for distinct $u, v \notin U_1$ either $|I(u) \cap I(v)| \geq \alpha/12$ or there is a matching of size at least $\alpha/12$ whose edges have one end in $I(u)$ and the other in $I(v)$; this follows from $I(u)$ and $I(v)$ being independent sets of size at least $2\alpha/3$.

- For every $u \in U_1$ and $xy \in M(u)$, add the ordered triples (u, x, y) and (u, y, x) to \mathcal{F} .
- For every $u \in U_2$, fix v, w such that $v \notin U_1$ and (uvw) is a triangle. Because $I(u)$ and $I(v)$ are two independent sets of size at least $2\alpha/3$. If $|I(u) \cap I(v)| \geq \alpha/12$, add the ordered triple (u, x, v) to \mathcal{F} for each $x \in I(u) \cap I(v)$. Otherwise there is a matching M of size at least $\alpha/12$ between $I(u)$ and $I(v)$. Add to \mathcal{F} the ordered quintuple (u, x, y, v, w) , for each edge $xy \in M$ with $x \in I(u)$ and $y \in I(v)$.
- For every $u \in U_3$, fix $v, w, x \neq u$ such that (vwx) is a triangle, $v \notin U_1$, and uw is an edge. If $|I(u) \cap I(v)| \geq \alpha/12$, add (u, y, v, x, w) with $y \in I(u) \cap I(v)$ to \mathcal{F} . Otherwise, let M be an $I(u) - I(v)$ matching of size at least $\alpha/12$, and add (u, y, z, v, w) to \mathcal{F} , for all $yz \in M$ (where $y \in I(u)$ and $z \in I(v)$).

Notice that for every $u \notin U_4$ there are at least $\alpha/12$ ordered tuples in \mathcal{F} whose first element is u , showing that $|\mathcal{F}| \geq 2n/3 \cdot \alpha/12 = \alpha n/18$. Moreover, for every two vertices x, y , there is at most one ordered tuple in \mathcal{F} whose first element is x and the second y .

Let X be a random set of vertices, obtained by including each vertex with probability $1/2$, independently. Let \mathcal{P} be the set of ordered pairs (x, y) such that $x, y \in X$ and \mathcal{F} contains a tuple whose first element is x and the second is y , and whose other elements are not in X . Then $\mathbb{E}(|\mathcal{P}|) \geq |\mathcal{F}| \cdot 2^{-5} \geq \alpha n/576$. Fix an outcome of X such that $|\mathcal{P}| \geq \alpha n/576$. Let H be the graph on vertex set X , with edges xy such that at least one of (x, y) and (y, x) is in \mathcal{P} . Then $e(H) \geq |\mathcal{P}|/2 \geq \alpha n/1152$. The graph H satisfies the requirements of the lemma. \square

7 Conclusion

In this paper we proved that if a graph G has sufficiently many vertices, and satisfies $\kappa(G) > \alpha(G)$, then G is pancyclic, thereby proving a long standing conjecture of Jackson and Ordaz [19] for large graphs. Of course, it would be nice to settle the conjecture for all graphs. As mentioned in the introduction, it is plausible that a slight generalisation of Jackson and Ordaz's conjecture holds, namely that every graph G that contains a triangle and satisfies $\kappa(G) \geq \alpha(G)$ is pancyclic. Tools from this paper are

likely to be helpful in proving this generalisation (if true), but not everything carries through directly, and we have decided not to pursue this direction as the paper is long enough as it is.

As a different avenue for potential future research, we mention a question from [12]. In [12], Draganić, Munhá-Correia, and Sudakov proved that if G is Hamiltonian and $n \geq (2 + \Omega(1))\alpha^2$, where n is the number of vertices in G and α is G 's independence number, then G is pancyclic. While this bound is tight, up to the $\Omega(1)$ error term, the authors suspect that a much weaker bound suffices for guaranteeing a cycle of length $n - 1$.

Question 7.1. *Is there a constant c such that if G is a Hamiltonian graph on n vertices, with independence number α where $n \geq c \cdot \alpha$, then G has a cycle of length $n - 1$?*

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