Rainbow Hamiltonicity in uniformly coloured perturbed graphs

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Abstract

We investigate the existence of a rainbow Hamilton cycle in a uniformly edge-coloured randomly perturbed graph. We show that for every $\delta \in (0,1)$ there exists $C = C(\delta) > 0$ such that the following holds. Let G_0 be an n-vertex graph with minimum degree at least δn and suppose that each edge of the union of G_0, with the random graph $\mathbf{G}(n, C/n)$ on the same vertex set, gets a colour in $[n]$ independently and uniformly at random. Then, with high probability, $G_0 \cup \mathbf{G}(n, C/n)$ has a rainbow Hamilton cycle.

This improves a result of Aigner-Horev and Hefetz, who proved the same when the edges are coloured uniformly in a set of $(1+\varepsilon)n$ colours.

1 Introduction

Given $\delta \in (0,1)$, let $\mathcal{G}_{\delta,n}$ be the collection of graphs on vertex set $[n]$ with minimum degree at least δn. Determining the minimum δ that guarantees that every member of $\mathcal{G}_{\delta,n}$ contains a given spanning subgraph is a central theme in extremal combinatorics. The prototypical example is Dirac's theorem [12], which says that the minimum δ such that every member of $\mathcal{G}_{\delta,n}$ is Hamiltonian is 1/2. On the other hand, one of the main pursuits of probabilistic combinatorics is understanding the minimum p such that $\mathbf{G}(n,p)$, the binomial random graph on $[n]$ with edge probability p, contains a given subgraph *with high probability*[1]. Following the breakthrough of Pósa [25], it was proven in [20, 21] that

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[1]We say that a sequence of events $(A_n)_{n\in\mathbb{N}}$ holds *with high probability* if $\mathbf{P}[A_n] \to 1$ as $n \to \infty$.

$\mathbf{G}(n,p)$ is Hamiltonian with high probability, if it has minimum degree at least 2 with high probability, showing that the threshold p for Hamiltonicity is $(1+o(1))\log n/n$.

As an interpolation between the two models, Bohman, Frieze and Martin [8] introduced the *perturbed graph* model. Given a fixed $\delta > 0$, this is defined as $G_0 \cup \mathbf{G}(n,p)$, where $G_0 \in \mathcal{G}_{\delta,n}$, i.e. this is the union of some graph on vertex set $[n]$ with minimum degree at least δn, and the random graph $\mathbf{G}(n,p)$ on the same vertex set. In [8] the authors showed that there exists C, depending only on δ, such that for all $G_0 \in \mathcal{G}_{\delta,n}$, the perturbed graph $G_0 \cup \mathbf{G}(n,C/n)$ is with high probability Hamiltonian. That is, for every graph with linear minimum degree, adding linearly many random edges results in a graph that is with high probability Hamiltonian. This is best possible for all $\delta \in (0,1/2)$ up to the value of C, since the complete bipartite graph with parts of size δn and $(1-\delta)n$ requires $\Omega(n)$ edges to be Hamiltonian. (When $\delta \geq 1/2$ no random edges are needed, due to Dirac's theorem.) By now there is a sizeable literature on the perturbed model; see e.g. [1,2,4,5,9,10,22,23].

In this paper we consider a rainbow variant of the above result. A subgraph H of an edge coloured graph G is called *rainbow* if no two edges of H share a colour. For a finite set of colours $\mathcal{C}$, a graph G is *uniformly coloured* in $\mathcal{C}$ if each edge of G gets a colour in $\mathcal{C}$ independently and uniformly at random. The problem of finding rainbow subgraphs of uniformly coloured graphs is well studied, in particular for $\mathbf{G}(n,p)$ [6, 11, 13, 14, 17]. The problem of finding rainbow subgraphs in the uniformly coloured perturbed graph $G \sim G_0 \cup \mathbf{G}(n,p)$, where $G_0 \in \mathcal{G}_{\delta,n}$, was first considered more recently [1–5]. In particular, the problem of containing a rainbow Hamilton cycle was first addressed by Anastos and Frieze [5], who showed that if the number of colours is at least about $120n$, then $G \sim G_0 \cup \mathbf{G}(n,C/n)$ has with high probability a rainbow Hamilton cycle, for C depending only on δ and all $G_0 \in \mathcal{G}_{\delta,n}$. Aigner-Horev and Hefetz [3] improved this result by showing that, at the same edge probability in the random graph, $n + o(n)$ colours suffice. We prove that the optimal number of colours suffices.

Theorem 1.1. *For any $\delta \in (0,1)$ there exists $C > 0$ such that the following holds. For $G_0 \in \mathcal{G}_{\delta,n}$, let $G \sim G_0 \cup \mathbf{G}(n,C/n)$ be uniformly coloured in $[n]$. Then with high probability G contains a rainbow Hamilton cycle.*

As explained above, our result has the optimal edge probability, up to the dependence of C on δ, for $\delta \in (0,1/2)$.

The paper is structured as follows. In Section 2 we sketch the proof of Theorem 1.1. In Section 3 we prove Theorem 1.1 assuming Lemma 3.1, the key lemma of the paper. Next in Section 4 we state and prove some preliminary results that we need. In Section 5 we prove the existence of 'gadgets' which underpin Lemma 3.1. In Section 6 we prove Lemma 3.1.

Throughout the paper, we will assume that n is sufficiently large. Asymptotic notation hides absolute constants: if for some $x, \varepsilon, n > 0$ we write $x = O(\varepsilon n)$, then there is an absolute constant $C > 0$, which does not depend on x, ε, n or any other parameters, such that $x \leq C\varepsilon n$. We write $x \ll y$ if $x < f(y)$ for an implicit positive increasing function f.

We denote the set of colours on the edges of a graph H by $\mathcal{C}(H)$, and say that a graph H is *spanning* in a colour set $\mathcal{C}'$ if $\mathcal{C}(H) = \mathcal{C}'$. Finally, we let $G_{\delta,n}$ be an arbitrary member of $\mathcal{G}_{\delta,n}$, which is the family of n-vertex graphs with minimum degree at least δn.

2 Proof sketch

Our proof uses the *absorption method*. This method is typically applicable when one searches for a spanning subgraph, and involves two stages: finding an almost spanning subgraph; and dealing with the remainder, by having a 'special' set of vertices, put aside at the beginning, that can cover any sufficiently small set of vertices.

This is done in Lemma 3.1, which says the following: with high probability there exists a rainbow path P_{abs} such that, for any sets of vertices V' and colours $\mathcal{C}'$ with $|V'| = |\mathcal{C}'|$, disjoint from the vertices and colors of P_{abs}, there exists another rainbow path Q with vertex set $V' \cup V(P_{\text{abs}})$ and colours $\mathcal{C}' \cup \mathcal{C}(P_{\text{abs}})$, whose ends can be any vertices in V'.

We now sketch the proof of Lemma 3.1. We first put aside a subset of the vertices and a subset of the colours, which are typically called the 'reservoir' (also called 'flexible set'), that have the following property: for any sets of vertices and colours $V', \mathcal{C}'$ of the same small size (much smaller than the reservoir) which are disjoint from the reservoir, we can find a rainbow path P_0 that uses V', $\mathcal{C}'$ and a $\Theta(|V'|)$ subset of the vertices and colours in the reservoir.

Then the question is how to cover the rest of the reservoir; to this end, we build an 'absorbing structure' (P_{abs} above) which has the following property: it can 'absorb' any subset of vertices and colours of the same size of the reservoir in a rainbow path P'_{abs}. Then combining P'_{abs} and P_0 gives Q.

The path P_{abs} and the 'absorbing structure' in which it resides are built by putting together several 'absorbing gadgets', graphs on $\Theta(1)$ vertices with the following property: each gadget has two paths with the same endpoints such that one avoids a designated pair of a vertex and a colour in the reservoir, and the other one 'absorbs' the same pair; see Figure 1. The construction of the gadgets is done in Section 5. This absorbing structure was introduced by Gould, Kelly, Kühn and Osthus [18] for constructing rainbow Hamilton paths in random optimal colourings of the complete graph, and is based on ideas of Montgomery [24].

For finding an almost spanning rainbow path we use a rainbow version of depth first search [3, 14], which was used for the same problem in [3].

3 Proof of Theorem 1.1

In this section we prove the main theorem, Theorem 1.1. We will use Lemma 3.1 below, which we prove in Section 6.

Lemma 3.1. *Let $\delta, \gamma, \eta \in (0,1)$ and $C > 0$ be constants such that $C^{-1} \ll \eta \ll \gamma \ll \delta$. Let $G \sim G_{\delta,n} \cup \mathbf{G}(n, C/n)$ be uniformly coloured in $\mathcal{C} = [n]$. Then, with high probability, G has a rainbow path P_{abs} of length at most γn with the following property. For any $V' \subseteq V \setminus V(P_{abs})$, $\mathcal{C}' \subseteq \mathcal{C} \setminus \mathcal{C}(P_{abs})$ with $2 \leq |V'| = |\mathcal{C}'| \leq \eta n$ and distinct $x, y \in V'$, there exists a path Q such that*

- *Q has ends x, y,*
- *$V(Q) = V(P_{abs}) \cup V'$,*
- *$\mathcal{C}(Q) = \mathcal{C}(P_{abs}) \cup \mathcal{C}'$.*

The next lemma is a rainbow version of a commonly used consequence of the depth first search algorithm [7], which we will use to find an almost spanning rainbow path. This lemma was used in [3] for the same problem.

Lemma 3.2 (Prop. 2.1 [3]; Lem. 2.17 [14])**.** *Let G be a graph with its edges coloured in a set $\mathcal{C}$. If for any two disjoint sets of vertices X, Y of size k we have $|\mathcal{C}(E(X,Y))| \geq |V(G)|$, then G has a rainbow path of length at least $|V(G)| - 2k + 1$.*

The next lemma can easily be proved using Chernoff's bound (cf. Theorem 4.1).

Lemma 3.3. *Let $\alpha \in (0,1)$ and $C > 0$ be constants with $C^{-1} \ll \alpha$. Let $G \sim \mathbf{G}(n, C/n)$ be uniformly coloured in $\mathcal{C} = [n]$. Then, with high probability, for any two disjoint sets of vertices X, Y of size αn we have $|\mathcal{C}(E(X,Y))| \geq (1-\alpha)n$.*

Our main theorem now follows easily.

Proof of Theorem 1.1. Let η, γ be constants such that $C^{-1} \ll \eta \ll \gamma \ll \delta$. By Lemmas 3.1 and 3.3 we may assume that there exists a path P_{abs} with the properties in Lemma 3.1; and that for any disjoint $X, Y \subseteq V$ of size $k = \eta n/4$ we have $|\mathcal{C}(E(X,Y))| \geq n - k$. Let $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}(P_{\text{abs}})$, let $V' \subseteq V \setminus V(P_{\text{abs}})$ be an arbitrary set of size k, and let $V_2 = V \setminus (V' \cup V(P_{\text{abs}}))$. So $|\mathcal{C}_2| \geq n - \gamma n$ and $|V_2| = |\mathcal{C}_2| - k - 1$. Then, for every disjoint $X, Y \subseteq V_2$ of size k,

$$|\mathcal{C}(E(X,Y)) \cap \mathcal{C}_2| \geq |\mathcal{C}_2| - k \geq |V_2|.$$

Therefore, in the spanning subgraph of $G[V_2]$ whose edges are edges in G coloured in $\mathcal{C}_2$, by Lemma 3.2 there exists a rainbow path P_2 of length at least $|V_2| - 2k + 1$. Hence $V_2' = V(G) \setminus (V(P_{\text{abs}}) \cup V(P_2))$ has size between k and $3k \leq \eta n - 2$ and $\mathcal{C}_2' = \mathcal{C} \setminus (\mathcal{C}(P_{\text{abs}}) \cup \mathcal{C}(P_2))$ has size $|V_2'| + 2$. Let x, y be the endpoints of P_2. Then by the property of P_{abs} there exists a rainbow path Q spanning in $V(P_{\text{abs}}) \cup V_2' \cup \{x, y\}$ and $\mathcal{C}(P_{\text{abs}}) \cup \mathcal{C}_2'$ with endpoints x, y. Then $P_2 \cup Q$ is a rainbow Hamilton cycle. $\square$

4 Preliminaries

In this section we collect three preliminary results that we need: the Chernoff bound, cf. Theorem 4.1; that random sparse subgraph of dense hypergraphs have large matchings, cf. Lemma 4.2; and that in the perturbed graph, between any two vertices, there is a large rainbow collection of paths of length three, cf. Lemma 4.3.

Theorem 4.1 (Chernoff Bound, [19, eq. (2.8) and Theorem 2.8])**.** *For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that the following holds. Let X be the sum of mutually independent indicator random variables and write $\mu = \mathbf{E}[X]$. Then*

$$\mathbf{P}[|X-\mu| \geq \varepsilon\mu] \leq 2\mathrm{e}^{-c_\varepsilon \mu}.$$

The next lemma, despite its technical appearance, proves the following straightforward statement: quite *sparse* random subgraphs of dense hypergraphs contain, with high probability, a matching of linear size.

Lemma 4.2. *Let $\alpha, c, c' > 0$ and $r \geq 2$ be an integer such that $c' \ll \alpha, c, r$. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with at least αn^r edges.*

Let $\mathcal{H}_m$ be the random subgraph of $\mathcal{H}$ that consists of $m = cn$ edges of $\mathcal{H}$, chosen with replacement and uniformly at random. Then, with probability at least $1 - \mathrm{e}^{-\frac{c\alpha n}{4}}$, the hypergraph $\mathcal{H}_m$ has a matching of size at least $c'n$.

Let $\mathcal{H}_p$ be the random subgraph of $\mathcal{H}$, where we keep each edge independently with probability $p = cn^{-r+1}$. Then with probability at least $1 - \mathrm{e}^{-\frac{c\alpha n}{2r}}$, the hypergraph $\mathcal{H}_p$ has a matching of size at least $c'n$.

Proof. Write $\beta(\mathcal{G})$ for the size of the largest matching of a hypergraph $\mathcal{G}$.

It is not hard to see that $\mathcal{H}$ contains an induced subgraph of minimum degree at least αn^{r-1}. Hence, without loss of generality, we may assume that $\mathcal{H}$ has minimum degree at least αn^{r-1}.

We first prove the result for $\mathcal{H}_m$. Suppose $\beta(\mathcal{H}_m) < c'n$, and let M be a maximal matching. Then $S = V(\mathcal{H}) \setminus V(M)$ is an independent set in $\mathcal{H}_m$ and $|S| \geq (1 - rc')n$. By the minimum degree condition of $\mathcal{H}$, the number of edges with all vertices in S is at least $\frac{1}{r}|S|(\alpha n^{r-1} - rc'n^{r-1}) \geq \frac{1}{r}(1-rc')\,(\alpha - c')n^r$. This gives

$$\begin{aligned}
\mathbf{P}[S \text{ is independent}] &= \left(1 - \frac{e(\mathcal{H}[S])}{e(\mathcal{H})}\right)^m \\
&\leq \exp\left(-cn \cdot \frac{\frac{1}{r}(1-rc')\,(\alpha - rc')n^r}{\binom{n}{r}}\right) \\
&\leq \exp\left(-\frac{1}{2}(r-1)!\,c\,(1-rc')\,(\alpha - rc')n\right)
\end{aligned}$$

Then, since $rc' < 1/2$, the number of $S \subseteq V$ with $|S| \geq (1 - rc')n$ is at most

$$n\binom{n}{(1-rc')n} = n\binom{n}{rc'n} \leq \mathrm{e}^{2rc'n}(rc')^{-rc'n}.$$

Thus by the union bound $\mathbf{P}\left[\beta(\mathcal{H}_m) < c'n\right] \leq \exp\left(f_{c,r,\alpha}(c')n\right)$, where

$$f_{c,r,\alpha}(c') = 2rc' - rc'\ln(rc') - \frac{(r-1)!}{2}c(1-rc')(\alpha - rc')$$

Since $f_{c,r,\alpha}(c')$ is continuous near 0 and $f_{c,r,\alpha}(c') \to 0 - 0 - \frac{(r-1)!}{2}\,c\,\alpha < 0$ as $c' \to 0$, for $c' = c'(c, r, \alpha)$ sufficiently small $f_{c,r,\alpha}(c') \leq -\frac{(r-1)!}{4}\,c\,\alpha \leq -\frac{1}{4}c\alpha$, which gives the first part of the lemma.

For the second part of the lemma observe that the same argument works: with S as above, in $\mathcal{H}_p$ we have

$$\mathbf{P}\left[S \text{ is independent}\right] = (1-p)^{e(\mathcal{H}[S])} \leq \exp\left(-cn^{-r+1} \cdot \frac{1}{r}(1-rc')\,(\alpha - rc')n^r\right)$$

and a similar calculation as above shows that the probability there is such an S is at most $\mathrm{e}^{-\frac{c\alpha n}{2r}}$. □

Lemma 4.3 (Triangles and Short Paths)**.** *Let $0 < \delta < 1$, $q, C > 0$ and $\rho, \lambda_{con} \ll \delta, q, C$. Let $\mathcal{C}$ be a set of colours of size qn. Let $G \sim G_{\delta,n} \cup \mathbf{G}(n, C/n)$ be uniformly coloured in $\mathcal{C}$. Then, with probability at least $1 - \mathrm{e}^{-\lambda_{con}n}$, the following holds. For any $u, v \in V(G)$ there is a matching M of size at least ρn such that the colours of the edges ux, xy, vy, for $xy \in M$, are all distinct.*

Proof. Fix $u, v \in V$. Let ρ_1 be a constant such that $\rho, \lambda_{\mathrm{con}} \ll \rho_1 \ll C, \delta, q$.

By the minimum degree assumption, there exist disjoint subsets $N_u \subseteq N_{G_{\delta,n}}(u)$, $N_v \subseteq N_{G_{\delta,n}}(v)$, of size $\delta n/2$. Consider the bipartite graph with bipartition (N_u, N_v) and edges

$$\left\{\, zw \in E(\mathbf{G}(n, C/n)) :\ z \in N_u,\ w \in N_v \right\}.$$

This is a random subgraph of the complete bipartite graph, with each part having order $\delta n/2$, and edge probability C/n. Hence, by Lemma 4.2, with probability $1 - \mathrm{e}^{-\Omega(C\delta n)}$, there is matching M of size $\rho_1 n$.

For each $zw \in M$, reveal whether the path $uzwv$ is rainbow, without exposing the colours. Then each $uzwv$ is rainbow independently with probability $1 - o(1)$. Hence by Chernoff's bound (Theorem 4.1), with probability $1 - \mathrm{e}^{-\Omega(\rho_1 n)}$, there is $M' \subseteq M$ with $|M'| \geq \rho_1 n/2$ such that each $uzwv$ is rainbow, for all $zw \in M'$.

Let $\mathcal{P} = \{uzwv : zw \in M'\}$. It remains to show we can find a large $M'' \subseteq M'$ such that the collection $\mathcal{P}' = \{uzwv : zw \in M''\}$ is rainbow.

Now reveal the colours on the edges in $\mathcal{P}$. By symmetry, each triple of distinct colours in $\mathcal{C}$ is equally likely to appear in $\mathcal{P}$. Hence $\mathcal{P}$ corresponds to selecting uniformly at random with replacement $|\mathcal{P}| \geq \rho_1 n/2$ edges from the complete 3-graph with vertex set $\mathcal{C}$. Thus, by Lemma 4.2, with probability $1 - \mathrm{e}^{-\Omega(\rho_1 qn)}$, there exists $M'' \subseteq M'$ of size ρn so that the colours of $\mathcal{P}' = \{uxyv : xy \in M''\}$ form a matching in the complete 3-graph on $\mathcal{C}$ i.e. $\mathcal{P}'$ is rainbow.

The probability this fails for some pair u, v is, by the union bound, at most

$$n^2 \cdot \left(\mathrm{e}^{-\Omega(C\delta n)} + \mathrm{e}^{-\Omega(\rho_1 n)} + e^{-\Omega(\rho_1 qn)}\right) \leq \mathrm{e}^{-\lambda_{\mathrm{con}} n},$$

proving the lemma. □

5 Finding absorbers

In this section we prove Lemma 5.2, which asserts that for any vertex v, colour c and any small (but linear in size) set of forbidden vertices and colours, we can find an 'absorber' (cf. Definition 5.1) for v, c. These absorbers are the building blocks for P_{abs} in Lemma 3.1. To construct these absorbers we will need to find a rainbow 4-cycle containing a given colour c, and none of the forbidden vertices and colours. This is the most technical part of our proof, and is done in Lemma 5.5.

Definition 5.1 (Absorber)**.** *Let v be a vertex and c a colour. A* (v, c)-absorber *is a graph $A_{v,c}$ with $v \in V(A_{v,c})$ and $c \in \mathcal{C}(A_{v,c})$ that has two paths P, P' with the following properties.*

- *They are rainbow.*
- *They have the same endpoints.*
- *P is spanning in $V(A_{v,c})$ and $V(P') = V(P) \setminus \{v\} = V(A_{v,c}) \setminus \{v\}$.*
- *P is spanning in $\mathcal{C}(A_{v,c})$ and $\mathcal{C}(P') = \mathcal{C}(P) \setminus \{c\} = \mathcal{C}(A_{v,c}) \setminus \{c\}$.*

We call P the (v, c)-absorbing path *and P' the* (v, c)-avoiding path*. The* internal vertices *of $A_{v,c}$ are $V(A_{v,c}) \setminus \{v\}$ and the* internal colours *are $\mathcal{C}(A_{v,c}) \setminus \{c\}$.*

For the sake of concreteness, we will refer to one of the endpoints of the paths as the first vertex *of the absorber and the other one as the* last vertex*.*

Lemma 5.2. *Let $0 < \delta < 1$, $C > 0$ and $C^{-1} \ll \nu \ll \delta$. Let $G \sim G_{\delta,n} \cup \mathbf{G}(n, C/n)$ be uniformly coloured in $\mathcal{C} = [n]$. Then with high probability the following holds. For any $v \in V(G)$ and $c \in \mathcal{C}$ and for all $V' \subseteq V(G)$ and $\mathcal{C}' \subseteq \mathcal{C}$ that have size at least $(1 - \nu)n$, there exists a (v, c)-absorber on 11 vertices with internal vertices in V' and internal colours in $\mathcal{C}'$.*

Our absorbers will consist of the union of a triangle, a 4-cycle and two paths of length three between opposite vertices of the cycle and between a vertex in the triangle and a vertex in the 4-cycle. We require the colours of the triangle to match the internal colours of the square. See Figure 1.

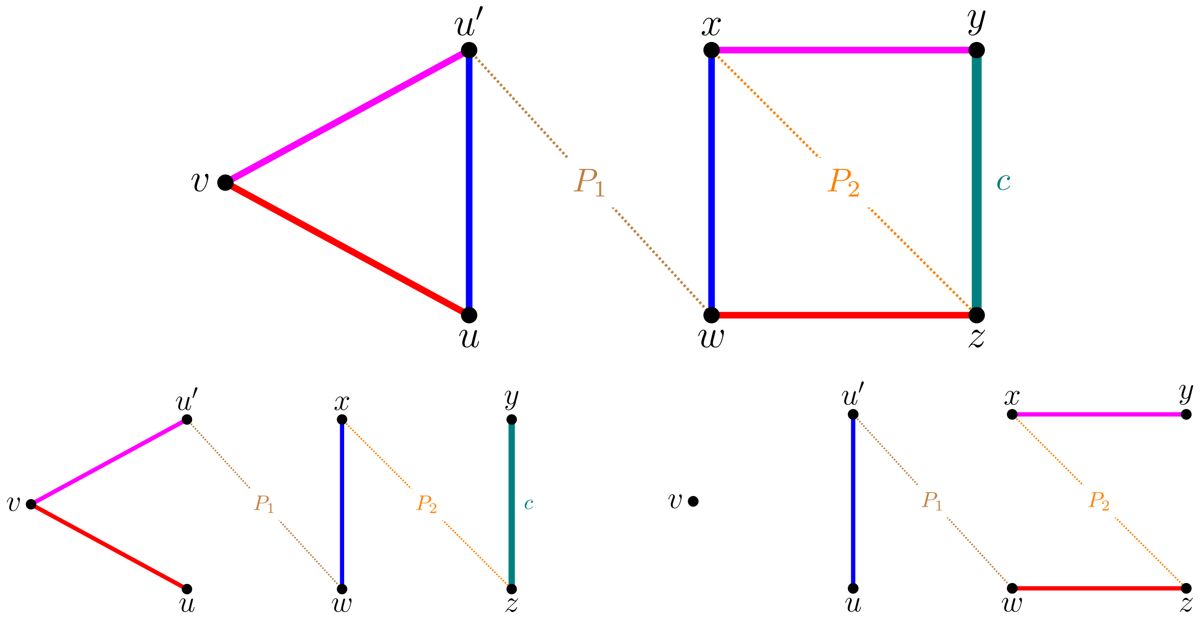

Figure 1: At the top is a (v,c)-absorber. At the bottom the first figure shows the (v,c)-absorbing path and the second figure the (v,c)-avoiding path.

5.1 Finding squares

We will use the following theorem, due to Fox and Sudakov [15], that is based on the dependent random choice method.

Theorem 5.3 (Theorem 3.1 [15]; see also Prop. 5.3 [16]). *Let $\alpha, \alpha' > 0$ be constants such that $\alpha' \ll \alpha$. Let G be a bipartite graph of order n with bipartition (A,B) and $e(A,B) \geq \alpha n^2$. Then there are $A' \subseteq A$, $B' \subseteq B$ such that for all $a \in A', b \in B'$, the number of paths of length three between a and b in $G[A',B']$ is at least $\alpha' n^2$.*

Lemma 5.4. *Let $\alpha, \beta > 0$ be constants such that $\beta \ll \alpha$. Let G be a bipartite graph on n vertices with bipartition (A,B) and $e(A,B) \geq \alpha n^2$. Then there exist disjoint sets $A_1, A_2 \subseteq A$, $B_1, B_2 \subseteq B$, such that for any $a \in A_1, b \in B_1$, the number of paths of length three between a, b with internal vertices in A_2, B_2 is at least βn^2. Moreover, the minimum degree of $G[A_1, B_1]$ is at least βn.*

Proof. Let α' satisfy $\beta \ll \alpha' \ll \alpha$ and let A', B' be given by Theorem 5.3. Let (A_1, A_2) be a random partition of A', and (B_1, B_2) be a random partition of B', i.e. each $a \in A'$ lies in A_1 independently with probability $1/2$, and similarly for B_1.

Then, since $G[A', B']$ has minimum degree at least $\alpha' n$, for each $a \in A'$ the expected number of neighbours of a in B_1 is at least $\alpha' n/2$; the same is true for the number of neighbours of $b \in B'$ in A_1. Hence from Chernoff's bound, for any $a \in A', b \in B'$,

$$\mathbf{P}\left[|N(a) \cap B_1| \geq \alpha' n/3\right], \mathbf{P}\left[|N(b) \cap A_1| \geq \alpha' n/3\right] \geq 1 - \mathrm{e}^{-\Omega(\alpha' n)}.$$

Consider a pair $a \in A', b \in B'$. Notice that the number of paths of length three between a, b in $G[A', B']$ is equal to the number of edges between $G[N(a), N(b)]$. From Theorem 5.3, the number of edges of $G[N(a) \cap B', N(b) \cap A']$ is at least $\alpha' n^2$, hence there are $B_{a,b} \subseteq N(a) \cap B'$, $A_{a,b} \subseteq N(b) \cap A'$ such that $G[A_{a,b}, B_{a,b}]$ has minimum degree at least $\alpha' n$. Then the expected number of neighbours of each $a' \in A_{a,b}$ in $B_{a,b} \cap B_2$ is at least $\alpha' n/2$, so by Chernoff's bound, for $a' \in A_{a,b}$,

$$\mathbf{P}\left[|N(a') \cap B_{a,b} \cap B_2| \geq \alpha' n/3\right] \geq 1 - \mathrm{e}^{-\Omega(\alpha' n)}.$$

Similarly, for $b' \in B_{a,b}$,

$$\mathbf{P}\left[|N(b') \cap A_{a,b} \cap A_2| \geq \alpha' n/3\right] \geq 1 - \mathrm{e}^{-\Omega(\alpha' n)}.$$

Hence, for each $a \in A', b \in B'$, the probability that the minimum degree of $G[A_{a,b} \cap A_2, B_{a,b} \cap B_2]$ is less than $\alpha' n/3$ is, by the union bound, at most

$$|A_{a,b}|\, \mathrm{e}^{-\Omega(\alpha' n)} + |B_{a,b}|\, \mathrm{e}^{-\Omega(\alpha' n)} \leq \mathrm{e}^{-\Omega(\alpha' n)}.$$

Moreover, the number of paths of length three between a, b with internal vertices in A_2, B_2 is

$$e(N(a) \cap B_2, N(b) \cap A_2) \geq e(N(a) \cap B_2 \cap B_{a,b}, N(b) \cap A_2 \cap A_{a,b}),$$

which is at least the square of the minimum degree of $G[N(a) \cap B_2 \cap B_{a,b}, N(b) \cap A_2 \cap A_{a,b}]$. Hence, the probability that the number of paths of length three between $a \in A', b \in B'$ with internal vertices in A_2, B_2 is less than $\alpha'^2 n^2/9$ is at most $\mathrm{e}^{-\Omega(\alpha' n)}$.

By the union bound over $a \in A', b \in B'$ and pairs $(a, b) \in A' \times B'$ the probability that a random partition fails to satisfy the lemma, with $\beta = \alpha'^2/9$, is at most $n\mathrm{e}^{-\Omega(\alpha' n)} + n^2 \mathrm{e}^{-\Omega(\alpha' n)} < 1$. Thus there exists a partition as desired. □

Lemma 5.5. *Let $\delta, q_1, q_2, \lambda_{sq}$ be constants such that $0 < \delta < 1$, $0 < q_2 < q_1$ and $0 < \lambda_{sq} \ll \delta, q_2$. Let $\mathcal{C}$ be a set of colors with $|\mathcal{C}| = q_1 n$ and $\mathcal{C}_0 \subseteq \mathcal{C}^3$ be a collection of colour triples that are pairwise disjoint, with $|\mathcal{C}_0| = q_2 n$. Let G be a graph of order n and minimum degree at least δn which is uniformly coloured in $\mathcal{C}$. Then, with probability at least $1 - \mathrm{e}^{-\lambda_{sq} n}$, the following holds. For any $c \in \mathcal{C}$ there exists a 4-cycle in G coloured (c_1, c_2, c_3, c), for some $(c_1, c_2, c_3) \in \mathcal{C}_0$.*

Proof. Let $\beta, \gamma_1, \gamma_3$ be constants such that $\beta \ll \delta$ and $\lambda_{\mathrm{sq}} \ll \gamma_3 \ll \gamma_1 \ll \gamma \ll \beta, q_1^{-1}$.

Fix $c \in \mathcal{C}$. By passing to a bipartite subgraph of G with at least $e(G)/2$ edges, from Lemma 5.4 there exist disjoint $A_1, B_1, A_2, B_2 \subseteq V(G)$ such that the bipartite graph $G[A_1, B_1]$ has minimum degree at least βn, and for all $a \in A_1, b \in B_1$, the number of edges in $G[N(a) \cap B_2, N(b) \cap A_2]$ is at least βn^2. We will reveal the colours of the edges in $G[A_1 \cup A_2, B_1 \cup B_2]$ in the order $E(A_1, B_1), E(A_1, B_2), E(A_2, B_1), E(A_2, B_2)$.

Since each edge of $G[A_1, B_1]$ is coloured c independently with probability $(q_1 n)^{-1}$, by Lemma 4.2, with probability at least $1 - \mathrm{e}^{-\Omega(\lambda_{\mathrm{sq}} n)}$, there exists a matching $M \subseteq G[A_1, B_1]$ of size at least γn with all edges coloured c.

For $(c_1, c_2, c_3) \in \mathcal{C}_0$ say an edge $e \in E(A_2, B_2)$ is *good* for (c_1, c_2, c_3), if, when $\mathcal{C}(e) = c_2$, it completes a 4-cycle coloured (c_1, c_2, c_3, c) with vertices in A_1, B_1, A_2, B_2 (in this order). Notice that, this definition does not depend on the colours of the edges in $G[A_2, B_2]$. Let

$$F(c_1, c_2, c_3) := \{e \in E(A_2, B_2) : e \text{ is good for } (c_1, c_2, c_3)\}.$$

Claim 5.6. *Fix $(c_1, c_2, c_3) \in \mathcal{C}_0$. With probability at least $1 - \mathrm{e}^{-\Omega(\lambda_{sq} n)}$, $|F(c_1, c_2, c_3)| \geq \gamma_3 n$.*

Proof. Let $ab \in M$. Since $e(N(a) \cap B_2, N(b) \cap A_2) \geq \beta n^2$, there exist $A_{ab} \subseteq N(b) \cap A_2$, $B_{ab} \subseteq N(a) \cap B_2$ such that $G[A_{ab}, B_{ab}]$ has minimum degree at least βn. Let G' be the spanning subgraph of G such that $xy \in E(G')$ if and only if the following holds:

- If $xy \in E_G(A_1, B_1)$ then $xy \in M$.
- If $xy \in E_G(A_1, B_2)$ then $x \in V(M) \cap A_1$ and $y \in B_{xM(x)}$, where $M(x)$ is the neighbour of x in M.
- If $xy \in E_G(A_2, B_1)$ then $y \in V(M) \cap B_1$ and $x \in A_{M(y)y}$.
- If $xy \in E_G(A_2, B_2)$ then $xy \in E_G(A_e, B_e)$ for some $e \in M$.

Since G is 4-partite with parts A_1, B_1, A_2, B_2, this exhausts all possible edges of G'.

Then the number of edges of $G'[A_1, B_2]$ is at least $\sum_{a \in A_1 \cap V(M)} |B_{ab}| \geq \gamma \beta n^2$. Moreover, each edge is coloured c_1 independently with probability $(q_1 n)^{-1}$. Therefore, by Lemma 4.2, with probability at least $1 - \mathrm{e}^{-\Omega(\lambda_{\mathrm{sq}} n)}$, there is a matching M_1 in $G'[A_1, B_2]$ coloured c_1 that has size at least $\gamma_1 n$.

Finally, we will find a large matching M_3 coloured c_3 which, along with M_1 and M will give us a large number of good edges for (c_1, c_2, c_3). To this end, let G'' be the spanning subgraph of G' such that $xy \in E(G'')$ if and only if the following holds:

- if $xy \in E_{G'}(A_1, B_1)$ then $xy \in M$ and $x \in V(M) \cap V(M_1)$.
- If $xy \in E_{G'}(A_1, B_2)$ then $xy \in M_1$.
- If $xy \in E_{G'}(A_2, B_1)$ then $y \in V(M) \cap B_1$, $M(y) \in V(M_1)$, and $x \in A_{M(y)y} \cap N_{G'}(M_1(M(y)))$.
- If $xy \in E_{G'}(A_2, B_2)$ then there exists $ab \in M$ such that $y = M_1(a)$ and $x \in A_{ab}$.

Again, this exhausts all possibilities for the edges of G''.

Then the number of edges of $G''[A_2, B_1]$ is at least

$$\sum_{ab \in M \colon a \in V(M_1) \cap A_1} |N_{G'}(M_1(a)) \cap A_{ab}| \geq \gamma_1 \beta n^2,$$

where we use that for all $ab \in M$ the minimum degree of $G''[A_{ab}, B_{ab}]$ is at least βn.

Moreover, each edge of $G''[A_2, B_1]$ is coloured c_3 independently with probability $(q_1 n)^{-1}$. Hence, by Lemma 4.2, with probability at least $1 - \mathrm{e}^{-\Omega(\lambda_{\mathrm{sq}} n)}$, there exists a matching M_3 in $G''[A_2, B_1]$ coloured c_3 that has size at least $\gamma_3 n$.

Let

$$F_0(c_1, c_2, c_3) := \{xy \in E_{G''}(A_2, B_2) : x \in V(M_3) \cap A_2\ \}.$$

Notice from the definition of G'' that every $x \in A_2$ has a neighbour in B_2, hence $|F_0(c_1, c_2, c_3)| \geq |M_3|$. Moreover, if $xy \in F_0(c_1, c_2, c_3)$, then, by the definition of G'', there are $a \in A_1, b \in B_1$ such that $ab \in M$, $ay \in M_1$, $xb \in M_3$; i.e. $\mathcal{C}(ab) = c$, $\mathcal{C}(ay) = c_1$, $\mathcal{C}(xb) = c_3$. Therefore, xy is a good edge for (c_1, c_2, c_3). Thus $F_0(c_1, c_2, c_3) \subseteq F(c_1, c_2, c_3)$, so $|F(c_1, c_2, c_3)| \geq |F_0(c_1, c_2, c_3)| \geq |M_3| \geq \gamma_3 n$ and the claim follows. $\square$

By the union bound over $(c_1, c_2, c_3) \in \mathcal{C}_0$, for which there are $q_2 n$ choices, Claim 5.6 implies that with probability at least $1 - \mathrm{e}^{-\Omega(\lambda_{\mathrm{sq}} n)}$, for each $(c_1, c_2, c_3) \in \mathcal{C}_0$, $|F(c_1, c_2, c_3)| \geq \gamma_3 n$. Let

$$F'(e) := \{c_2 \in \mathcal{C} : \text{there exist } c_1, c_3 \text{ such that } (c_1, c_2, c_3) \in \mathcal{C}_0 \text{ and } e \in F(c_1, c_2, c_3)\}.$$

Then, using that no two triples in $\mathcal{C}_0$ share a colour we have

$$\sum_{e \in E(A_2, B_2)} |F'(e)| = \sum_{(c_1, c_2, c_3) \in \mathcal{C}_0} |F(c_1, c_2, c_3)| \geq |\mathcal{C}_0|\, \gamma_3 n = q_2 \gamma_3 n^2.$$

Now we reveal the colours of $E(A_2, B_2)$. For $e \in E(A_2, B_2)$ let A_e be the event that e gets a good colour i.e. $\mathcal{C}(e) \in F'(e)$. Then $\mathbf{P}[A_e] = |F'(e)| / q_1 n$. Each edge is coloured independently, so the events A_e are mutually independent. Hence, the probability that no $e \in E(A_2, B_2)$ gets a good colour is

$$\begin{aligned}\prod_{e \in E(A_2, B_2)} (1 - \mathbf{P}[A_e]) &\leq \exp\left(- \sum_{e \in E(A_2, B_2)} \mathbf{P}[A_e]\right) \\ &= \exp\left(- \sum_{e \in E(A_2, B_2)} \frac{|F'(e)|}{q_1 n}\right) \leq \exp\left(-\frac{q_2 \gamma_3 n}{q_1}\right).\end{aligned}$$

Hence, with probability at least $1 - \mathrm{e}^{-\frac{q_2 \gamma_3 n}{q_1}}$, at least one edge gets a good colour, i.e. there exists $e \in E(A_2, B_2)$ such that $\mathcal{C}(e) \in F'(e)$, as required for the lemma.

The above fails for some colour c with probability at most

$$q_1 n \mathrm{e}^{-\Omega(\lambda_{\mathrm{sq}} n)} \leq \mathrm{e}^{\lambda_{\mathrm{sq}} n},$$

proving the lemma. $\square$

5.2 Proof of Lemma 5.2

Proof of Lemma 5.2. Let ρ, ρ' be constants satisfying $C^{-1} \ll \nu \ll \lambda, \rho, \rho' \ll \delta$. Fix $v \in V(G)$, $c \in \mathcal{C}$ and $V' \subseteq V(G)$, $\mathcal{C}' \subseteq \mathcal{C}$ of size at least $(1-\nu)n$.

For the next claim, it is useful to refer to Figure 1.

Claim 5.7. *With probability $1 - \mathrm{e}^{-\Omega(\lambda n)}$, there exist a 4-cycle $K = xyzw$ and a triangle $T = vuu'$ in $G[V']$ such that $\mathcal{C}(yz) = c$, $\mathcal{C}(xw) = \mathcal{C}(uu')$, $\mathcal{C}(xy) = \mathcal{C}(vu')$, $\mathcal{C}(zw) = \mathcal{C}(vu)$.*

Proof. Let $(V_\triangle, V_\square)$ be a random partition of V'. Then from Chernoff's bound, a union bound over $v \in V'$, and $\nu \ll 1$, with probability $1 - \mathrm{e}^{-\Omega(\delta n)}$, the graphs $G_{\delta,n}[V_\triangle]$, $G_{\delta,n}[V_\square]$ have minimum degree at least $\delta n/3$ and $|V_\triangle|, |V_\square| \geq n/3$.

First reveal the random edges and colours of $G[V_\triangle]$. Then, by Lemma 4.3, with probability $1 - \mathrm{e}^{-\Omega(\lambda n)}$, there is a collection Δ_v of ρn rainbow triangles that pairwise intersect only on v; are pairwise colour-disjoint; and $V(\Delta_v) \subseteq V_\triangle \cup \{v\}$. Let $\mathcal{C}_v$ be the collection of colour triples $(\mathcal{C}(vu), \mathcal{C}(uu'), \mathcal{C}(vu'))$ with $vuu' \in \Delta_v$, whose three colours are in $\mathcal{C}'$. Then $|\mathcal{C}_v| \geq \rho n - 3\nu n \geq (\rho/2)n$.

Next reveal the colours of edges in $G[V_\square]$. By setting $\mathcal{C}_0 = \mathcal{C}_v$ in Lemma 5.5, it follows that with probability $1 - \mathrm{e}^{-\Omega(\lambda n)}$ there exists a 4-cycle $xyzw$ and a triangle $vuu' \in \Delta_v$ with colours in $\mathcal{C}_v$, such that $\mathcal{C}(yz) = c$, $\mathcal{C}(xw) = \mathcal{C}(uu')$, $\mathcal{C}(xy) = \mathcal{C}(vu')$, $\mathcal{C}(wz) = \mathcal{C}(vu)$.

We fail to find a triangle or square as required with probability at most $\mathrm{e}^{-\Omega(\lambda n)}$. $\square$

By Lemma 4.3, with probability $1 - \mathrm{e}^{-\Omega(\lambda n)}$, for every $u, v \in V'$ there are $\rho' n$ rainbow paths of length three between u, v which are pairwise colour disjoint and internally vertex disjoint. Hence, with probability $1 - \mathrm{e}^{-\Omega(\lambda n)}$, this and the conclusion of Claim 5.7 hold simultaneously.

Then, using $\nu \ll \rho'$, there exists two colour- and vertex-disjoint rainbow paths P_1, P_3 of length 3 such that: P_1 has endpoints u_2, w; P_3 has endpoints x, z; the interiors of P_1, P_2 are in $V' \setminus (V(K) \cup V(C))$; and the colours of P_1, P_2 are in $\mathcal{C}' \setminus (\mathcal{C}(K) \cup \mathcal{C}(T))$. Then the graph $A_{v,c}$, defined as

$$A_{v,c} = K \cup T \cup P_1 \cup P_2,$$

is a (v, c)-absorber: the (v, c)-absorbing path is $uvu'P_1wxP_2zy$ and the (v, c)-avoiding path is $uu'P_1wzP_2xy$, and it is straightforward to check they satisfy Definition 5.1. Clearly $A_{v,c}$ has 11 vertices.

The number of $V' \subseteq V$ of size at least $(1-\nu)n$ is at most $n\binom{n}{\nu n} = \mathrm{e}^{O(\nu \log \nu)n}$, and the same bound holds for the number of $\mathcal{C}' \subseteq \mathcal{C}$ of the same size. Using $\nu \ll \lambda$, the probability we fail to find an absorber for some v, c, V', $\mathcal{C}'$ is by the union bound at most

$$n^2 \mathrm{e}^{O(\nu \log \nu)n} \cdot \mathrm{e}^{-\Omega(\lambda n)} \leq n^{-2}.$$

$\square$

6 Proof of Lemma 3.1

To cover an arbitrary small subset of the vertices using P_{abs} into a rainbow path Q we need the following lemma, which asserts that for any two vertices and a color we can connect them with a short rainbow path through a random subset of the vertices.

Lemma 6.1 (Flexible sets)**.** *Let $\zeta, \mu, \delta \in (0,1)$ and $C > 0$ be constants such that $C^{-1} \ll \zeta \ll \mu \ll \delta$. Let $G \sim G_{\delta,n} \cup \mathbf{G}(n, C/n)$. Then there exist $V_{flex} \subseteq V$, $\mathcal{C}_{flex} \subseteq \mathcal{C}$ of size $2\mu n$ such that with high probability the following holds. For all $u, v \in V$, $c \in \mathcal{C}$, and $V'_{flex} \subseteq V_{flex}$, $\mathcal{C}'_{flex} \subseteq \mathcal{C}_{flex}$ of size at least $(2\mu - \zeta)n$, there exists a rainbow path of length seven with endpoints u, v, internal vertices in V'_{flex} and colours in $\mathcal{C}'_{flex} \cup \{c\}$, that contains the colour c.*

Proof. Let γ be a constant satisfying $C^{-1} \ll \zeta \ll \gamma \ll \mu \ll \nu \ll \delta$.

For a colour c, let M_c be a largest matching of colour c in G, and for distinct vertices u, v, let $\mathcal{P}_{u,v}$ be a largest collection of pairwise vertex- and colour-disjoint rainbow paths of length three with endpoints u, v. By Lemmas 4.2 and 4.3, with probability $1 - e^{-\gamma n}$, we have $|M_c| \geq \gamma n$ and $|\mathcal{P}_{u,v}| \geq \gamma n$ for every colour c and distinct vertices u, v.

Let V' be a random subset of V, obtained by including each vertex independently with probability μ, and let $\mathcal{C}'$ be a random subset of $\mathcal{C}$, obtained by including each colour independently with probability μ.

Then, by Chernoff and union bounds, with high probability, the following properties hold.

- $|V'|, |\mathcal{C}'| \leq 2\mu n$,
- at least $\frac{1}{2}\mu^2\gamma n$ edges in M_c have both endpoints in V', for every $c \in \mathcal{C}$,
- at least $\frac{1}{2}\mu^5\gamma n$ paths in $\mathcal{P}_{u,v}$ have their interior vertices in V' and all colours in $\mathcal{C}'$, for all distinct $u, v \in V$.

Suppose that all three properties hold, and let V_{flex} be a subset of V that contains V' and has size $2\mu n$ and let $\mathcal{C}_{\text{flex}}$ be a subset of $\mathcal{C}$ that contains $\mathcal{C}'$ and has size $2\mu n$.

We show that these sets satisfy the requirements of the lemma. Indeed, fix u, v, c and $V'_{\text{flex}}, \mathcal{C}'_{\text{flex}}$ as in the lemma. Then, as $\zeta \ll \mu, \gamma$, there is an edge $e = xy \in M_c$ with both ends in V'_{flex}. Similarly, there are paths $P_1 \in \mathcal{P}_{u,x}, P_2 \in \mathcal{P}_{y,v}$ that are vertex- and colour-disjoint, their interiors are in V'_{flex}, and their colours are in $\mathcal{C}'_{\text{flex}} \setminus \{c\}$. Then $P_1 \cup e \cup P_2$ is a path that satisfies the requirements of the lemma. □

We will put together several (v, c)-absorbers to construct the paths in Lemma 3.1, by having a (v, c)-absorber for each edge of a bipartite graph which has the following property. This follows an idea introduced by Montgomery [24], which was adapted to the rainbow setting by Gould, Kelly, Kühn and Osthus [18].

Definition 6.2 (Def. 3.3, [18]). *Let H be a balanced bipartite graph with bipartition (A, B). We say H is* robustly matchable *with respect to A', B', for some $A' \subseteq A$ and $B' \subseteq B$ of equal size, if for every pair of sets $X \subseteq A', Y \subseteq B'$ with $|X| = |Y| \leq |A'|/2$, there is a perfect matching in $H[A \setminus X, B \setminus Y]$. We call A', B' the* flexible sets *of H.*

Proposition 6.3 (Lemma 4.5, [18]). *For every large enough $m \in \mathbb{N}$, there exists a 256-regular bipartite graph with bipartition (A, B) and $|A| = |B| = 7m$, which is robustly matchable with respect to some $A' \subseteq A, B' \subseteq B$ with $|A'| = |B'| = 2m$.*

Proof of Lemma 3.1. Let $\zeta, \mu, \nu \in (0, 1)$ be constants such that

$$C^{-1} \ll \eta \ll \zeta \ll \mu \ll \nu \ll \delta.$$

Let $V_{\text{flex}}, \mathcal{C}_{\text{flex}}$ be the sets given by Lemma 6.1 that have size $2\mu n$. By the union bound, the conclusions of Lemmas 4.3, 5.2 and 6.1 hold simultaneously with high probability. Assume they all hold.

Let $V_{\text{buf}}, \mathcal{C}_{\text{buf}}$ be arbitrary subsets of $V \setminus V_{\text{flex}}, \mathcal{C} \setminus \mathcal{C}_{\text{flex}}$ of size $5\mu n$. Let H be a bipartite graph on $(V_{\text{flex}} \cup V_{\text{buf}}, \mathcal{C}_{\text{flex}} \cup \mathcal{C}_{\text{buf}})$ that is isomorphic to a graph as in Proposition 6.3 such that $V_{\text{flex}}, \mathcal{C}_{\text{flex}}$ are the flexible sets.

Claim 6.4. *There is collection of absorbers $A_{v,c}$ on 11 vertices and rainbow paths $P_{v,c}$ of length three, for each edge vc in H, with the following properties: the internal vertices of $A_{v,c}$ and of $P_{v,c}$ are pairwise disjoint and disjoint of $V_{flex} \cup V_{buf}$; the internal colours of $A_{v,c}$ and the colours of $P_{v,c}$ are pairwise disjoint and disjoint of $\mathcal{C}_{flex} \cup \mathcal{C}_{buf}$; and for some ordering of the edges of H, the path $P_{v,c}$ starts with the last vertex of $A_{v',c'}$ and ends with the first vertex of $A_{v,c}$, where $v'c'$ is the predecessor of vc in the ordering (so we can ignore P_{vc} for the first edge vc).*

Proof. Let H_0 be a maximal subgraph of H with some ordering of its edges, for which we can find a collection of absorbers and paths as in the claim. Suppose for contradiction $H_0 \neq H$ and let $v_1c_1 \in E(H \setminus H_0)$ and v_0c_0 be the last edge of H_0 in the ordering, that has absorber A_{v_0,c_0}.

Let $V_0, \mathcal{C}_0$ be the union of the vertices and colours spanned by the absorbers for $E(H_0)$, the paths connecting them, and $V_{\text{flex}} \cup V_{\text{buf}}, \mathcal{C}_{\text{flex}} \cup \mathcal{C}_{\text{buf}}$. Then, since each absorber has 11 vertices and each path connecting consecutive absorbers has 4 vertices, we have

$$|V_0|, |\mathcal{C}_0| = O(e(H_0)) = O(\mu n) < \nu n/2,$$

where for the inequality we used that $\mu \ll \nu$. Hence by Lemma 5.2 there exists a (v_1, c_1)-absorber A_{v_1,c_1} on 11 vertices with internal vertices and internal colours disjoint from V_0 and $\mathcal{C}_0$. Moreover, by Lemma 4.3 there exists a rainbow path P_{v_1,c_1} of length three between the last vertex of A_{v_0,c_0} and the first vertex of A_{v_1,c_1}, with internal vertices disjoint from $V_0 \cup V(A_{v_1,c_1})$ and colours disjoint from $\mathcal{C}_0 \cup \mathcal{C}(A_{c_1,c_1})$. Then the subgraph of

H with edges $E(H_0) \cup \{v_1c_1\}$ satisfies the conditions of the claim and properly contains H_0, contradicting the maximality of H_0. $\square$

We can now define P_{abs}. Since H is regular bipartite, it has a perfect matching M. For $vc \in E(H)$, let $P_M(vc)$ be the (v,c)-absorbing path of $A_{v,c}$, if $vc \in E(M)$, and the avoiding path otherwise. Let $P_{vc} = \emptyset$ if vc is the first edge, and otherwise let P_{vc} be as in Claim 6.4. Set

$$P_{\text{abs}} = \bigcup_{vc \in E(H)} (P_M(vc) \cup P_{vc}).$$

Then P_{abs} uses each $v \in V_{\text{flex}} \cup V_{\text{buf}}$ and $c \in \mathcal{C}_{\text{flex}} \cup \mathcal{C}_{\text{buf}}$ precisely once; any other vertex and colour in P_{abs} is also used, by construction, precisely once. Therefore P_{abs} is a rainbow path that is spanning in $\bigcup_{vc\in E(H)}(V(A_{vc}) \cup V(P_{vc}))$ and $\bigcup_{vc\in E(H)}(\mathcal{C}(A_{vc}) \cup \mathcal{C}(P_{vc}))$ with endpoints the first vertex w of the first absorber and the last vertex w' of the last absorber.

We will now show how to construct Q, given $V' \subseteq V \setminus V(P_{\text{abs}})$ and $\mathcal{C}' \subseteq \mathcal{C} \setminus \mathcal{C}(P_{\text{abs}})$ of size between 2 and ηn, with endpoints $x, y \in V'$. Let $c_0 \in \mathcal{C}'$. From Lemma 6.1, there exists a rainbow path Q_1 with endpoints w, x, internal vertices in V_{flex} and colours in $\mathcal{C}_{\text{flex}} \cup \{c_0\}$, which includes the colour c_0 and has length 7.

Claim 6.5. *There exists a rainbow path Q_2 between w', y, with internal vertices $V''_{\text{flex}} \cup (V' \setminus x)$, and colours $\mathcal{C}''_{\text{flex}} \cup (\mathcal{C}' \setminus c_0)$, for some $V''_{\text{flex}} \subseteq V_{\text{flex}} \setminus V(Q_1)$, $\mathcal{C}''_{\text{flex}} \subseteq \mathcal{C}_{\text{flex}} \setminus \mathcal{C}(Q_1)$ with $|V''_{\text{flex}}| = |\mathcal{C}''_{\text{flex}}| \leq \mu n - 7$.*

Proof. The Claim will follow by applying greedily Lemma 6.1 to cover $V' \setminus x$, $\mathcal{C}' \setminus c_0$ using $V_{\text{flex}} \setminus V(Q_1)$, $\mathcal{C}_{\text{flex}} \setminus \mathcal{C}(Q_1)$ in a rainbow path with endpoints w' and y.

Fix a linear order of $V' \setminus x$ with y the last vertex. Let P_0 be a longest path from w' to a vertex in $V' \setminus x$ among all rainbow paths that start at w' and satisfy the following: if $u, v \in V(P_0) \cap (V' \setminus x)$ and $u < v$, then u appears before v on P_0; every seventh vertex on P_0 lies in $V' \setminus x$, and all other vertices are in $\{w'\} \cup V_{\text{flex}} \setminus V(Q_1)$; between consecutive vertices in $V' \setminus x$, and between w' and the first vertex in $V' \setminus x$, there is exactly one edge with colour in $\mathcal{C}' \setminus c_0$, and all other edges have colours in $\mathcal{C}_{\text{flex}} \setminus \mathcal{C}(Q_1)$.

Let z be the last vertex of P_0. If $z = y$ we are done so suppose otherwise, and let $z' \in V' \setminus x$ be the vertex after z in the order. Since, by construction, $|V(P_0) \cap V'| = |\mathcal{C}(P_0) \cap \mathcal{C}'|$, there is also $c_1 \in \mathcal{C}' \setminus \mathcal{C}(P_0)$.

Let $V'_{\text{flex}} = V_{\text{flex}} \setminus (V(P_0) \cup V(Q_1))$, $\mathcal{C}'_{\text{flex}} = \mathcal{C}_{\text{flex}} \setminus (\mathcal{C}(P_0) \cup \mathcal{C}(Q_1))$. Since $|P_0| \leq 7\,|V'| \leq 7\eta n$, and Q_1 has length 7, using $\eta \ll \zeta$, it follows that $|V'_{\text{flex}}| = |\mathcal{C}'_{\text{flex}}| \geq (2\mu - \zeta)n$. Hence by Lemma 6.1 there is a rainbow path P_1 between z, z' of length 7, that contains an edge with colour c_1, and whose internal vertices and other colours are in $V'_{\text{flex}}, \mathcal{C}'_{\text{flex}}$. Then $P_1 \cup P_0$ contradicts the maximality of P_0.

Let $V''_{\text{flex}} = V(P_0) \cap V_{\text{flex}}$, $\mathcal{C}''_{\text{flex}} = \mathcal{C}(P_0) \cap \mathcal{C}_{\text{flex}}$. Then $|V''_{\text{flex}}| = |\mathcal{C}''_{\text{flex}}| < |P_0| \leq \zeta n < \mu n - 7$, so we can take $Q_2 = P_0$. $\square$

Let $V_{\text{flex}}''' = (V(Q_1) \cup V(Q_2)) \cap V_{\text{flex}}$ and $\mathcal{C}_{\text{flex}}''' = (\mathcal{C}(Q_1) \cup \mathcal{C}(Q_2)) \cap \mathcal{C}_{\text{flex}}$. Then we have $|V_{\text{flex}}'''| = |\mathcal{C}_{\text{flex}}'''| \leq \mu n$. Hence by choice of H there is a matching M' between $V_{\text{flex}} \setminus V_{\text{flex}}'''$ and $\mathcal{C}_{\text{flex}} \setminus \mathcal{C}_{\text{flex}}'''$.

As before, for $vc \in E(H)$ let $P_{M'}(vc)$ be the (v,c)-absorbing path of $A_{v,c}$ if $vc \in E(M')$ and the avoiding path otherwise. Let

$$P_{\text{abs}}' = \bigcup_{vc \in E(H)} (P_{M'}(vc) \ \cup P_{vc}).$$

Then P_{abs}' is a rainbow path that is spanning in $V(P_{\text{abs}}) \setminus V_{\text{flex}}'''$ and $\mathcal{C}(P_{\text{abs}}) \setminus \mathcal{C}_{\text{flex}}'''$ with endpoints w, w'. Therefore $Q = Q_1 \cup P_{\text{abs}}' \cup Q_2$ is a rainbow path, spanning in $V(P_{\text{abs}}) \cup V'$ and $\mathcal{C}(P_{\text{abs}}) \cup \mathcal{C}'$ and has endpoints x, y. $\square$

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