PATH RAMSEY NUMBER FOR RANDOM GRAPHS

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Abstract. Answering a question raised by Dudek and Prałat, we show that if $pn \to \infty$, w.h.p., whenever $G = G(n,p)$ is 2-edge-coloured there is a monochromatic path of length $n(2/3 + o(1))$. This result is optimal in the sense that $2/3$ cannot be replaced by a larger constant.

As part of the proof we obtain the following result. Given a graph G on n vertices with at least $(1-\varepsilon)\binom{n}{2}$ edges, whenever G is 2-edge-coloured, there is a monochromatic path of length at least $(2/3 - 110\sqrt{\varepsilon})n$. This is an extension of the classical result by Gerencsér and Gyárfás which says that whenever K_n is 2-coloured there is a monochromatic path of length at least $2n/3$.

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1. Introduction

Considering the richness of Ramsey theory and the great interest in random graphs, it is natural to consider Ramsey properties of random graphs. The study of random Ramsey theory has proved particularly useful in the establishment of upper bounds on the size Ramsey number. For graphs G, F, H, we write $G \to (F, H)$ if for every red-blue colouring of the edges of G, there is either a red F or a blue H. If F, H are isomorphic, we use instead the notation $G \to H$. The size Ramsey number, denoted by $\hat{r}(H)$ is defined to be $\hat{r}(H) = \min\{|E(G)| : G \to H\}$.

Denote by P_n the path on n vertices. In [2], disproving a conjecture of Erdős [8], Beck showed that $\hat{r}(P_n) \le 900n$. In [5] Bollobás noted a slightly better bound, and recently Dudek and Prałat [7] gave an elementary proof of the bound $\hat{r}(P_n) \le 137n$. In fact, they proved that w.h.p., $G(n, \alpha/n) \to P_{\beta n}$ for suitable constants α, β.

Dudek and Prałat [7] also showed that w.h.p. $G(n,p) \to P_{(1/3+o(1))n}$ when $pn \to \infty$ and raised the question of determining the maximum l such that $G(n,p) \to P_l$ for $pn \to \infty$. Inspired by the result of Gerencsér and Gyárfás [9] which says that $K_n \to P_{2n/3}$, they asked if $G(n,p) \to P_l$ for $l = (2/3 + o(1))n$. Our main result answers this question in the affirmative.

Theorem 1. *Let $0 < p = p(n) < 1$ satisfy $pn \to \infty$. Then w.h.p. $G(n,p) \to P_{(2/3+o(1))n}$.*

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This result is essentially best possible since there is a 2-colouring of the edges of K_n with no monochromatic path on more than $\lfloor 2n/3+1 \rfloor$ vertices. To see this, divide the vertex set of K_n into two sets A, B such that $|A| = \lfloor n/3 \rfloor$, let the edges spanned by B be coloured red and colour the other edges blue (see Figure 1).

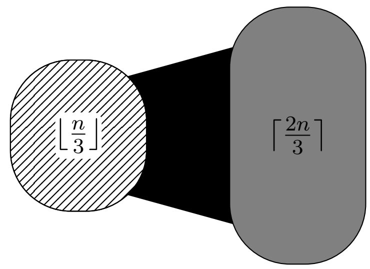

FIGURE 1. A black and grey graph on n vertices (the shaded part may be coloured arbitrarily) with no monochromatic path on more than $\lfloor \frac{2n}{3}+1 \rfloor$ vertices.

In fact, Gerencsér and Gyárfás [9] proved the following more general result.

Theorem 2. *Let $n \geq k + \lfloor (l+1)/2 \rfloor$. Then $K_n \to (P_{k+1}, P_{l+1})$.*

In order to prove Theorem 1, we extend Theorem 2 to graphs with a large number of edges.

Theorem 3. *Let $0 \leq \varepsilon \leq 1/64$, let $k \geq l$ and let G be a graph on $n \geq k+\lfloor (l+1)/2 \rfloor + 240\sqrt{\varepsilon}k$ vertices with at least $(1-\varepsilon)\binom{n}{2}$ edges. Then $G \to (P_{k+1}, P_{l+1})$.*

In particular, given $0 \leq \varepsilon \leq 1/64$, for every graph G on n vertices and at least $(1-\varepsilon)\binom{n}{2}$ edges, $G \to P_k$ where $k = \lfloor \frac{2n}{3} \rfloor - 110\sqrt{\varepsilon}n$.

Theorem 3 is a consequence of the following similar result, in which we consider graphs with large minimum degree rather than a large number of edges.

Theorem 4. *Let $0 < \varepsilon \leq 1/4$, let $k \geq l$ and let G be a graph on $n \geq k + \lfloor (l+1)/2 \rfloor + 100\varepsilon k$ vertices with minimum degree at least $(1-\varepsilon)n$. Then $G \to (P_{k+1}, P_{l+1})$.*

It is easy to deduce Theorem 3 from Theorem 4. By an averaging argument, it suffices to prove the assertion for $n = k + \lfloor (l+1)/2 \rfloor + 240\sqrt{\varepsilon}k$. By removing at most $\sqrt{\varepsilon}n$ vertices, we obtain a graph on $n' \geq (1-\sqrt{\varepsilon})n$ vertices and minimum degree at least $(1-2\sqrt{\varepsilon})n \geq (1-2\sqrt{\varepsilon})n'$ vertices. One can check that $(1-\sqrt{\varepsilon})n \geq k+\lfloor (l+1)/2 \rfloor + 200\sqrt{\varepsilon}k$, so the assertion of Theorem 3 follows from Theorem 4, with $2\sqrt{\varepsilon}$ in place of ε. For the second part, it is easy to check that when $k = l = \lfloor \frac{2n}{3} \rfloor - 110\sqrt{\varepsilon}n$, it follows that $n \geq k + \lfloor (l+1)/2 \rfloor + 240\sqrt{\varepsilon}k$.

In our proofs we shall use the following result which was proved independently by Dudek and Prałat [7] and Pokrovskiy [12].

Lemma 5. *For every graph G there exist two disjoint subsets $U, W \subseteq V(G)$ of equal size such that there are no edges between them and $G \setminus (U \cup W)$ has a Hamilton path.*

This result is the main tool used which was used by Dudek and Prałat in [7] to prove the bound $\hat{r}(P_n) \leq 137n$. It turns out that their proof may be modified to give a better upper bound, as stated in the following theorem.

Theorem 6. *For n sufficiently large, $\hat{r}(P_n) \leq 91n$.*

The rest of the paper is organised as follows. In Section 2 we give the proof of Lemma 5, as well as an easy but useful corollary. In Section 3 we give a short proof of a weaker version of Theorem 1 as well as the proof of the improved upper bound in Theorem 6.

We prove Theorem 4 in Section 4. In order to prove Theorem 1, we use the so-called sparse Regularity Lemma, due to Kohayakawa [11] and Rödl (see [6]). In Section 5 we state this result as well as some necessary notation. We prove Theorem 1 in Section 6 and finish with some concluding remarks in Section 7. Throughout the paper we omit floor and ceiling signs whenever they do not affect the arguments.

2. A useful lemma

In the proof of Theorems 1 and 4 we use the following lemma, which was obtained independently by Dudek and Prałat [7] and Pokrovskiy [12]. For the sake of completeness, we prove it here.

Lemma (5)**.** *For every graph G there exist two disjoint subsets $U, W \subseteq V(G)$ of equal size such that there are no edges between them and $G \setminus (U \cup W)$ has a Hamilton path.*

Proof. In order to find sets with the desired properties, we apply the following algorithm, maintaining a partition of $V(G)$ into subsets U, W and a path P. Start with $U = V(G), W = \emptyset$ and P an empty path. At each stage of the algorithm, do the following. If $|U| \leq |W|$, stop. Otherwise, if P is empty, move a vertex from U to P (note that $U \neq \emptyset$). If P is non-empty, let v be its endpoint. If v has a neighbour u in U, put u in P, otherwise move v to W.

Note that at any given point in the algorithm there are no edges between U and W. Furthermore, the value $|U| - |W|$ is positive at the beginning of the algorithm and decreases by one at every stage, thus at some point the algorithm will stop and will produce sets U, W with the required properties. □

Occasionally it is easier to use the following immediate consequence of Lemma 5.

Corollary 7. *Let G be a balanced bipartite graph on n vertices with bipartition $\{V_1, V_2\}$ which has no path of length k. Then there exist disjoint subsets $X_i \subseteq V_i$ such that $|X_1| = |X_2| \geq (n-k)/4$ and $e(G[X_1, X_2]) = 0$.*

Proof. By Lemma 5, there exist disjoint subsets $U, W \subseteq V(G)$ of equal size such that $e(G[U,W]) = 0$ and $V(G) \setminus (U \cup W)$ has a Hamilton path P. Note that P must alternate between V_1 and V_2 and has an even number of vertices, implying that $|V_1 \cap V(P)| = |V_2 \cap V(P)|$. It follows that $|U_1| + |W_1| = |U_2| + |W_2|$ where $U_i = U \cap V_i$ and $W_i = W \cap V_i$. Since $|U| = |W|$, we conclude that $|U_1| = |W_2|$ and $|U_2| = |W_1|$. Without loss of generality, suppose that $|U_1| \geq |U_2|$. Then $|U_1| = |W_2| \geq (n - |V(P)|)/4 \geq (n-k)/4$. Take $X_1 = U_1$ and $X_2 = W_2$.

□

3. An improved upper bound on the size Ramsey number for paths

Before we turn to the proofs of Theorems 4 and 1, we demonstrate how Lemma 5 alone can be used to obtain results about the path Ramsey number of random graphs. We start by proving the following weaker version of Theorem 1, using only elementary tools.

Lemma 8. *Let $0 < p = p(n) < 1$ and assume that $pn \to \infty$. Then w.h.p., $G(n,p) \to P_l$ for some $l = (1/2 + o(1))n$.*

In the proof of Lemma 8 we use the following easy consequence of Corollary 7.

Corollary 9. *Let G be a graph on n vertices such that $G \nrightarrow P_{k+1}$. Then there exist disjoint subsets $X, Y \subseteq V(G)$ of size at least $(n-2k)/4$ such that $e(G[X,Y]) = 0$.*

Proof. Consider a red-blue colouring of G with no monochromatic P_{k+1}. By Lemma 5, there exist disjoint sets U, W, both of size at least $(n-k)/2$ with no red edges between them. Considering the graph $G[U,W]$, it follows from Corollary 7 that there exist sets $X \subseteq U, Y \subseteq W$ of size at least $(n-2k)/4$, with no blue edges between them. We conclude that there are no edges of G between X and Y. □

We are now ready for the proof of Lemma 8, which is a relaxed version of Theorem 1.

Proof of Lemma 8. Let $G = G(n,p)$, where $np \to \infty$. Given $\alpha > 0$, suppose that $G \nrightarrow P_{(1/2-\alpha)n}$. By Corollary 9, there exist disjoint subsets $X, Y \subseteq V(G)$ of size at least $\alpha n/2$ with no edges of G between them. But this is a contradiction, since w.h.p., every two disjoint sets of at least $\alpha n/2$ vertices in G have an edge between them. It follows that for every $\alpha > 0$ w.h.p. $G(n,p) \to P_{(1/2-\alpha)n}$. □

Corollary 9 can be used to obtain an improvement over the upper bound $\hat{r}(P_n) \leq 137n$, which was obtained by Dudek and Prałat [7].

Theorem (6)**.** *For n sufficiently large, $\hat{r}(P_n) \leq 91n$.*

We note that our proof is very similar to the proof in [7], the main difference is our use of Corollary 9. We shall use the following lemma.

Lemma 10. *Let $c = 4.86$, $d = 7.7$ and $G = G(cn, d/n)$. Then w.h.p. the following two conditions hold.*

(1) $|E(G)| \leq (1 + o(1))\frac{c^2 d}{2} n$.
(2) *For every two disjoint sets $U, W \subseteq V(G)$ of size at least $\frac{c-2}{4} n$, we have $e(G[U, W]) > 0$.*

Proof. The number of edges in G is a binomial random variable with mean $\binom{cn}{2} \cdot \frac{d}{n} = (1 + o(1))\frac{c^2 d}{2} n$. Condition (1) follows from the concentration of binomial random variables around their mean.

We prove Condition (2) by the first moment method. Let Z denote the number of pairs (U, W) of disjoint subsets of $V(G)$ of size $\frac{c-2}{4} n$ with $e(G[U, W]) = 0$. The expectation of Z satisfies the following, where $\alpha = \frac{c-2}{4}$.

$$\mathbb{E}[Z] = \binom{cn}{\alpha n}\binom{(c-\alpha)n}{\alpha n}(1 - \frac{d}{n})^{(\alpha n)^2} \leq \frac{(cn)!}{((\alpha n)!)^2((c - 2\alpha)n)!}\exp(-d\alpha^2 n) \leq \exp(\beta n).$$

By Stirling's formula (stating that $n! = (1 + o(1))\sqrt{2\pi n}(n/e)^n)$, we can take

$$\beta = (c \log c - 2\alpha \log \alpha - (c - 2\alpha)\log(c - 2\alpha) - d\alpha^2) \leq -0.0005.$$

It follows that $\mathbb{E}[Z] \to 0$, implying that w.h.p. $Z = 0$, hence Condition (2) holds. □

Remark. The constants c, d in Lemma 10 were chose so as to minimise the number of edges in G under Condition (2).

The proof of Theorem 6 follows easily from Lemma 10.

Proof. Pick $c = 4.86$ and $d = 7.7$ as in Lemma 10 and denote $G = G(cn, \frac{d}{n})$. If $G \nrightarrow P_n$ then by Corollary 9 there exists disjoint subsets $X, Y \subseteq V(G)$ of size at least $(c-2)n/4$ such that $e(G[X, Y]) = 0$, contradicting Condition (2) from Lemma 10. We conclude that $G \to P_n$ w.h.p.. By Condition (1), we have that $|E(G)| \leq 91n$ w.h.p.. It follows that $\hat{r}(P_n) \leq 91n$ for large enough n.

□

4. Path Ramsey number for dense graphs

Before turning to the proof of Theorem 4, we remind the reader of its statement.

Theorem (4)**.** *Let $0 < \varepsilon < 1/4$, let $k \geq l$ and let G be a graph on $n \geq k + \lfloor (l+1)/2 \rfloor + 100\varepsilon k$ vertices with minimum degree at least $(1 - \varepsilon)n$. Then $G \to (P_{k+1}, P_{l+1})$.*

Proof. We start by proving Theorem 4 under the assumption that $k < (1/2 - \varepsilon)n$. If there is no red P_{k+1} by Lemma 5 there exist disjoint sets W_1, W_2 of size at least $(n-k)/2$ with no red edges between them. Since G has minimum degree at least $n(1-\varepsilon)$, we can greedily find a blue path on at least the following number of vertices, implying the existence of a blue P_{k+1}.

$$|W_1| + |W_2| - 2\varepsilon n \geq n - k - 2\varepsilon n = 2(1/2 - \varepsilon)n - k > k.$$

Hence, we can assume from now on that $k \geq (1/2 - \varepsilon)n \geq n/4$, so every vertex in G has at most $4\varepsilon k$ non neighbours. Putting $\delta = 4\varepsilon$, we have $n \geq k + \lfloor (l+1)/2 \rfloor + 25\delta k$. Furthermore, we can assume that $\delta k \geq 2$, otherwise G is a complete graph and Theorem 4 follows directly from Theorem 2.

The proof proceeds by induction. Clearly, the assertion of Theorem 4 holds if $k = 1$, so we may assume that $k \geq 2$. If $k > l$ then by induction there is either a red P_k or a blue P_{l+1}; in the latter case we are done. If $k = l$ then by induction there is either a red or a blue P_k. Thus, without loss of generality, there is a red path on k vertices which we denote by $P = (v_1 \dots v_k)$. Let $U = V(G) \setminus V(P)$. We consider three cases.

Case 1: $G[U]$ contains a blue path Q on $13\delta k$ vertices.

Let Q_1 be a maximal path extending Q by alternating between vertices of P and U and which has both ends in U. Let $U' = U \setminus V(Q_1)$ and $V' = V(P) \setminus V(Q_1)$. Let Q_2 be a maximal path alternating between U' and V' which has both ends in U'. Denote the ends of Q_i by x_i, y_i, for $i = 1, 2$. We show that $|V(Q_1)| + |V(Q_2)| \geq l + 3\delta k$.

Suppose this is not the case. In particular, since $|U| \geq l/2 + 25\delta k$, the paths Q_1 and Q_2 do not cover U. Pick a vertex $z \in U \setminus (V(Q_1) \cup V(Q_2))$. Note that all but at most $3\delta k$ vertices of P are adjacent to all of x_1, x_2, z. By our assumption on the lengths of Q_1 and Q_2, the number of vertices of P which are in Q_1 or Q_2 is at most $(1/2 - 5\delta)k$, hence there exist vertices v_i, v_{i+1} which are adjacent to all of x_1, x_2, z. We assume that v_i and v_{i+1} have no common red neighbours in x_1, x_2, z because otherwise we obtain a red P_{k+1}. It follows that without loss of generality, v_i is joined in blue to two of x_1, x_2, z, contradicting the maximality of Q_1 and Q_2.

Let Q_2' be a subpath of Q_2 with ends $x_2', y_2' \in U$ satisfying $|V(Q_2')| + |V(Q_1)| = l + 3\delta k$. Similarly to the above, there exist vertices v_i, v_{i+1} which are both neighbours of all of x_1, y_1, x_2', y_2'. By the maximality of Q_1, none of v_i, v_{i+1} is adjacent in blue to one of x_1, y_1 and to one of x_2', y_2'. Furthermore, we may assume that v_i and v_{i+1} have no common red neighbour in $\{x_1, y_1, x_2', y_2'\}$. Thus, without loss of generality, x_1, y_1 are blue neighbours of v_i and x_2', y_2' are blue neighbours of v_{i+1}. Denote by C_1 and C_2 the blue cycles obtained by adding v_i to Q_1 and v_{i+1} to Q_2, and let $U_i = V(C_i) \cap U$.

We assume that there is no blue P_{l+1}. It follows that $|V(C_1)| \leq l$, implying that $|U_2| \geq \frac{3}{2}\delta k$, and that there are no blue edges between C_1 and C_2.

The number of vertices in $V(P) \setminus (V(C_1) \cup V(C_2))$ is at least $(1/2 + 3\delta)k$, hence there exists j such that $v_j, v_{j+1} \notin V(C_1) \cup V(C_2)$. Note that none of v_j and v_{j+1} can have blue neighbours in both U_1 and U_2, because otherwise we obtain a blue P_{l+1}. Also, we can assume that v_j, v_{j+1} have no red common neighbour in either U_1 or U_2, because otherwise there is a red P_{k+1}. Thus, recalling that v_j, v_{j+1} have at most δk non neighbours in G, without loss of generality, v_j is joined in red to all but at most δk vertices of U_1 and v_{j+1} is joined in red to all but at most δk vertices in U_2.

Let $w_2 \in U_2$ be any red neighbour of v_{j+1} (such a vertex exists because $|U_2| \geq \frac{3}{2}\delta k$). Since w_2 is connected to all but at most δk vertices of U_1 and these edges must all be red, U_1 contains a vertex w_1 which is a red neighbour of both w_2 and v_j (such a vertex exists because $|U_1| \geq |V(Q_1)| \geq 13\delta k$). We obtain a red path $(v_1 \dots v_j w_1 w_2 v_{j+1} \dots v_k)$ on $k+2$ vertices. This finishes the proof of Theorem 3 in the first case.

Case 2: $l \leq (1-16\delta)k$.

Let Q_1 be a maximal blue path alternating between U and P with both ends in U. Similarly, let Q_2 be a maximal blue path with both ends in U, alternating between $U \setminus V(Q_1)$ and $V(P) \setminus V(Q_1)$. As in the previous case, it can be shown that $|V(Q_1)| + |V(Q_2)| \geq l + 6\delta k$.

Let Q_2' be a subpath of Q_2 such that $|V(Q_1)| + |V(Q_2')| = l + 6\delta k$. As before, there exists j such that $v_j, v_{j+1} \notin V(Q_1) \cup V(Q_2')$ and they are joined in G to all ends of the two paths. The vertices v_j, v_{j+1} can be used to extend Q_1, Q_2' into blue vertex disjoint cycles C_1, C_2, whose sum of lengths is $l + 6\delta k$ and each of which has length at least $6\delta k$. The proof of Theorem 4 can now be finished as in the first case.

Case 3: $l \geq (1-16\delta)k$ and $G[U]$ contains no blue path of length at least $13\delta k$.

We conclude from Lemma 5 that there exist two disjoint sets $W_1, W_2 \subseteq U$ with no blue edges between them of equal size satisfying the following inequality.

$$|W_1| = |W_2| \geq \frac{1}{2}(|U| - 13\delta k) = \frac{1}{2}(l/2 + 12\delta k) \geq \frac{1}{2}(1/2 + 4\delta)k.$$

Since every vertex in G is adjacent to all but at most δk vertices, we can greedily find a red path Q in U such that the following holds.

$$|V(Q)| \geq |W_1| + |W_2| - 2\delta k = (1/2 + 2\delta)k.$$

Let X be the set of the first and last $(1/4+\delta)k$ vertices of P. We assume that there is no red edge between X and Q, because otherwise there is a red P_{k+1}. Note that $|V(Q)| \geq |X| \geq (1/2 + 2\delta)k$, hence we may greedily construct a blue path alternating between X and $V(Q)$ on at least the following number of vertices.

$$2|X| - 2\delta k \geq (1 + 2\delta)k \geq l + 1.$$

Hence there exists a blue P_{l+1}, completing the proof of Theorem 4. □

5. Sparse Regularity Lemma

We shall make use of a variant of Szemerédi's Regularity Lemma [13] for sparse graphs, often referred to as the sparse Regularity Lemma, which was proved independently by Kohayakawa [11] and Rödl (see [6]). Before stating the theorem, we introduce some notation.

Given two disjoint sets of vertices U, V in a graph, we define the density $d_p(U, V)$ of edges between U and V with respect to p to be

$$d_p(U,V) = \frac{e(G[U,V])}{p|U||V|}, \tag{1}$$

where $e(G[U,V])$ is the number of edges between U and V. We say that a bipartite graph with bipartition U, V is (ε, p)-regular if for every $U' \subseteq U, V' \subseteq V$ with $|U'| \geq \varepsilon|U|, |V'| \geq \varepsilon|V|$ the density $d_p(U', V')$ satisfies $|d_p(U', V') - d_p(U, V)| \leq \varepsilon$.

Given a graph G, a partition $\{V_1, \ldots, V_t\}$ of $V(G)$ is called an (ε, p)-regular partition if it is an equipartition (i.e. the sizes of the sets differ by at most one), and if all but at most ε of the pairs V_i, V_j induce an (ε, p)-regular graph.

Given $0 < \eta, p < 1, D \geq 1$, a graph G is called (η, p, D)-upper-uniform if for all disjoints sets of vertices U_1, U_2 of size at least $\eta|V(G)|$, the density $d_p(U_1, U_2)$ is at most D. Note that random graphs are w.h.p. upper uniform (with suitable parameters).

We are now ready to state the sparse Regularity Lemma of Kohayakawa and Rödl.

Theorem 11. *For every $\varepsilon > 0$, t and $D > 1$ there exist $\eta > 0$ and T such that for every $0 \leq p \leq 1$, every (η, p, D)-upper-uniform graph admits an (ε, p)-regular partition into s parts where $t \leq s \leq T$.*

We shall use a variant of Theorem 11, namely the coloured version of the sparse Regularity Lemma.

Theorem 12. *For every $\varepsilon > 0$, t, l and $D > 1$ there exist $\eta > 0$ and T such that for every $0 \leq p \leq 1$, if $G_1, \ldots, G_l$ are (η, p, D)-upper-uniform graphs on vertex set V, there is an equipartition of V into s parts, where $t \leq s \leq T$, for which all but at most ε of the pairs induce a regular pair in each G_i.*

6. Path Ramsey number for random graphs

We are now ready to prove Theorem 1, whose statement is as follows.

Theorem (1). *Let $0 < p = p(n) < 1$ satisfy $pn \to \infty$. Then w.h.p. $G(n, p) \to P_{(2/3+o(1))n}$.*

Proof. Let $0 < p < 1$ be such that $pn \to \infty$ and let $\alpha > 0$. We show that w.h.p., for every 2-edge-colouring of $G = G(n,p)$ there is a monochromatic path of length at least $(2/3 - \alpha)n$.

Pick $\varepsilon > 0$ small and t large (taking $t = 1/\varepsilon$ and $\varepsilon \le 1/64$ small enough such that $110\sqrt{\varepsilon}+7\varepsilon \le \alpha$ would do). Let η, T be the constants arising from the application of Theorem 12 with $\varepsilon, t, l = 2, D = 2$. Note that w.h.p., for every two disjoint subsets $U, W \subseteq V(G)$ of size at least ηn, we have

$$1/2 \le d_p(U,W) \le 2. \tag{2}$$

In particular, G is w.h.p. $(\eta, p, 2)$-upper-uniform. Thus, by Theorem 12, given a 2-edge-colouring of G, there exists an (ε, p)-regular partition $V_1, \ldots, V_s$ with $t \le s \le T$. By (2), we may assume that $d_p(V_i, V_j) \ge 1/2$ for every $1 \le i < j \le s$.

Let H be the auxiliary graph with vertex set $[s]$ where ij is an edge iff V_i, V_j induce a regular bipartite graph in both red and blue. We colour an edge ij in H red if the red density $d_p(V_i, V_j)$ is at least $1/4$ and blue otherwise (so if ij is blue, the blue density is at least $1/4$).

Since the partition $V_1, \ldots, V_s$ is (ε, p)-regular, the number of edges in H is at least $(1-\varepsilon)\binom{s}{2}$. It follows from Theorem 3 that H contains a monochromatic path P on at least $l = (2/3-\delta)s$ vertices, where $\delta = 110\sqrt{\varepsilon}$ (assuming $\varepsilon > 0$ is small enough). Denote by $i_1, \ldots, i_l$ the vertices of P.

Assuming without loss of generality that P is red, we show that G contains a red path of length at least $(2/3 - \alpha)n$. We divide each set V_{i_j} into two sets U_j, W_j of equal sizes, so $|U_j| = n/2s$. Let P_j be a longest red path in the bipartite graph $G[U_j, W_{j+1}]$. The following claim shows that P_j covers most vertices in $U_j \cup W_{j+1}$.

Claim 13. *For every $1 \le j \le l$, P_j covers at least $1 - 4\varepsilon$ of the vertices of $U_j \cup W_{j+1}$.*

Proof. Suppose that for some j, P_j covers at most $1-4\varepsilon$ of the vertices of $U_j \cup W_{j+1}$. Set $U = U_j$ and $W = W_{j+1}$. By Corollary 7, there exist sets $X \subseteq U, Y \subseteq W$ with $|X| = |Y| \ge \varepsilon|U|$, such that there are no red edges between X and Y. But by the regularity of the partition $V_1, \ldots, V_s$, the density $d_p(U,V)$ is within ε of the density of red edges between U and W, which is at least $1/4$. In particular, G has a red edge between X and Y, contradicting our assumption, so Claim 13 holds. □

We now show that the paths $P_1, \ldots, P_{l-1}$ can be joined to a path Q while losing only a few of the vertices. Let X_j be the set of first $2\varepsilon|V_1|$ vertices of P_j and similarly let Y_j be the set of last $2\varepsilon|V_1|$ vertices of P_j. Since the paths P_j alternate between the sets U_j, W_{j+1}, we have that $|Y_j \cap V_{i_j}|, |X_{j+1} \cap V_{i_{j+1}}| \ge \varepsilon|V_1|$. It follows from the fact that $i_j i_{j+1}$ is a red edge in H that there is a red edge between Y_j and X_{j+1}. Hence G has a red path Q which contains all vertices of $V(P_1) \cup \ldots \cup V(P_{l-1})$ but at most $4\varepsilon|V_1|(l-1)$.

By Claim 13, we have that $|P_j| \geq (1-4\varepsilon)|V_1|$, so the following holds.

$$|Q| \geq (1-8\varepsilon)(l-1)|V_1| = (1-8\varepsilon)(s(2/3-\delta)-1)\cdot\frac{n}{s} \geq$$
$$(2/3-(\delta+1/t+6\varepsilon))n \geq (2/3-\alpha)n.$$

This completes the proof of Theorem 1.

□

7. Concluding Remarks

We remark that stronger versions of Theorem 4 for the symmetric case $k=l$ were proved by Benevides, Łuczak, Scott, Skokan and White [3] and by Gyárfás and Sárközy [10]. The results in [3] imply in particular that for every $\varepsilon > 0$, there exists n_0 such that for every graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq 3n/4$ satisfies $G \to P_{(2/3-\varepsilon)n}$. The condition on the minimum degree is best possible. The proofs in [3] and [10] rely heavily on the regularity lemma, whereas our proof of Theorem 4 is elementary.

It may be interesting to strengthen Theorem 4 so as to prove a similar result to the aforementioned result of Benevides et al. [3] in the non diagonal case, namely when k and l are not necessarily equal. Furthermore, it may be interesting to obtain the result of Benevides et al. [3] using elementary methods.

Finally, we note that the gap between the best known lower and upper bounds on the size Ramsey number is still very wide. Theorem 6 gives an upper bound of $\hat{r}(P_n) \leq 91n$, which is to our knowledge the best known upper bounds. Bollobás [4] proved the best know lower bound of $\hat{r}(P_n) \geq (1+\sqrt{2})n-2$, improving Beck's result [1] who showed that $\hat{r}(P_n) \geq \frac{9}{4}n$. It would be very interesting to try to close this gap.

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