THE SIZE-RAMSEY NUMBER OF 3-UNIFORM TIGHT PATHS

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Abstract. Given a hypergraph H, the size-Ramsey number $\hat{r}_2(H)$ is the smallest integer m such that there exists a hypergraph G with m edges with the property that in any colouring of the edges of G with two colours there is a monochromatic copy of H. We prove that the size-Ramsey number of the 3-uniform tight path on n vertices $P_n^{(3)}$ is linear in n, i.e., $\hat{r}_2(P_n^{(3)}) = O(n)$. This answers a question by Dudek, Fleur, Mubayi, and Rödl for 3-uniform hypergraphs [*On the size-Ramsey number of hypergraphs*, J. Graph Theory **86** (2016), 417–434], who proved $\hat{r}_2(P_n^{(3)}) = O(n^{3/2} \log^{3/2} n)$.

§1. Introduction

For hypergraphs G and H and an integer s, we denote by $G \to (H)_s$ the property that in any s-colouring of the edges of G there is a monochromatic copy of H. The *s-colour* size-Ramsey number $\hat{r}_s(H)$ is

$$\hat{r}_s(H) := \min\{|E(G)| : G \to (H)_s\}.$$

Erdős [10] asked if $\hat{r}_2(P_n) = O(n)$, which was answered positively by Beck [2] using the probabilistic method. An explicit construction for the same results was given by Alon and Chung [1]. Many successive improvements led to the currently best known bounds $5n/2 - 15/2 \leqslant \hat{r}_2(P_n) \leqslant 74n$ (see, e.g., [2, 5, 9] for lower bounds, and [2, 8, 9, 13] for upper bounds). For $s \geqslant 2$ colours, Dudek and Prałat [9] and Krivelevich [12] proved that there are constants c and C such that $cs^2 n \leqslant \hat{r}_s(P_n) \leqslant Cs^2(\log s)n$.

The systematic investigation of size-Ramsey questions for hypergraphs was initiated by Dudek, Fleur, Mubayi, and Rödl [7]. Besides cliques and trees, they studied generalisations of paths.

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We say that an r-uniform hypergraph is an *ℓ-path* if there exists an ordering of its vertices such that every edge is composed of r consecutive vertices, two (vertex-wise) consecutive edges share exactly ℓ vertices, and every vertex is contained in an edge. For $1 \leqslant \ell \leqslant r-1$, let $P_{n,\ell}^{(r)}$ denote the *r-uniform ℓ-path* on n vertices and for the *tight path*, where $\ell = r-1$, we write $P_n^{(r)}$. Dudek, Fleur, Mubayi, and Rödl [7] deduced from Beck's result [2] that $\hat{r}_2(P_{n,\ell}^{(r)}) = O(n)$, when $1 \leqslant \ell \leqslant r/2$. Furthermore, they proved that $\hat{r}_2(P_n^{(r)}) = O_r(n^{r-1-\alpha}\log^{1+\alpha} n)$, where $\alpha = (r-2)/(\binom{r-1}{2}+1)$.

This was improved and extended to more colours by Lu and Wang [14], who showed that $\hat{r}_s(P_n^{(r)}) = O_r(s^r(n\log n)^{r/2})$ for $s \geqslant 2$ colours. Dudek, Fleur, Mubayi, and Rödl [7] asked if $\hat{r}_2(P_n^{(r)}) = O_r(n)$ for $r \geqslant 3$. We answer this question for 3-uniform hypergraphs by proving the following result.

Theorem 1. *The* 2*-colour size-Ramsey number of the* 3*-uniform tight path is*

$$\hat{r}_2(P_n^{(3)}) = O(n).$$

Trivially, we need at least n edges, so this is asymptotically optimal. As observed in [7], bounds on size-Ramsey numbers for some uniformity can be used to obtain bounds for larger uniformities. We obtain the following corollary.

Corollary 2. *For any integer r such that $3 \mid r$, the* 2*-colour size-Ramsey number of the r-uniform $(2r/3)$-path is*

$$\hat{r}_2(P_{n,2r/3}^{(r)}) = O(n).$$

To see this, take the graph given by Theorem 1 and replace every vertex by a set of $r/3$ vertices. Then each 3-edge naturally gives an r-edge, and every 3-uniform tight path becomes an r-uniform $(2r/3)$-path.

Our proof combines new ideas and the method developed by Clemens, Jenssen, Kohayakawa, Morrison, Mota, Reding, and Roberts [6] for estimating the size-Ramsey number of powers of paths (see also [4, 11]). It is plausible that ideas from [4, 11] may provide a strategy to solve the case with $s \geqslant 3$ colours. However, the question whether the size-Ramsey number of a tight path is linear for hypergraphs with uniformity $r \geqslant 4$ remains open and requires additional ideas.

§2. Preliminaries

In this short section, we give a sketch of our proof of Theorem 1 and state two simple lemmas about random graphs.

2.1. **Sketch of the proof of Theorem 1.** We will first sketch a proof for $\hat{r}_2(P_n) = O(n)$. It is not hard (cf. Lemmas 3 and 4 below) to obtain a graph G with $O(n)$ edges such that for any two sufficiently large and disjoint sets of vertices A and B there is a path of length n alternating between A and B. Given such a graph G, we show that $G \to (P_n)_2$. Consider an arbitrary 2-colouring of the edges of G with colours blue and red. If there is no blue P_n in G we can show (cf. Lemma 6 below) that there are two sets A and B of size at least n with no blue edges in between. By the property of G mentioned above there exists a P_n alternating between A and B, which unequivocally has to be red.

For the proof of Theorem 1 we follow, in principle, the same strategy. Based on a blow-up of a power of a similar graph G, we define a 3-uniform hypergraph H and claim that $H \to (P_n^{(3)})_2$. We define an auxiliary (generalised) graph F on $V(G)$, which has 2- and 3-edges, such that a long path in F gives a blue $P_n^{(3)}$ in H. If F does not contain a long path, then we find a family of disjoint sets such that no edge of F lies between these sets (cf. Lemma 6). Then by the properties of G there exists a path in G alternating through these sets. As there are no edges of F 'interfering' with this path, we are able to turn it into a red $P_n^{(3)}$ in H.

In the next section we provide the lemmas needed to obtain G. Afterwards, in Section 3 we introduce the notion of $(2,3)$-graphs, which, as can be seen above, plays a key role in our argument. Finally, we prove Theorem 1 in Section 4.

2.2. **Sparse graphs with many long paths.** The following two lemmas are proved in [6]. Basically, together they imply that for every k and n there exists a graph G with $O_k(n)$ edges such that, for any disjoint sets of vertices $A_1, \ldots, A_{k+1}$ that are large enough, there exists a path of length n 'alternating' through these sets.

Lemma 3 ([6, Lemma 3.1])**.** *For every pair of positive constants ε and a, there is a constant b such that, for any large enough n, there is a graph H with $v(H) = an$ and $\Delta(H) \leqslant b$ such that the following holds:*

(P1$_n$) *For every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geqslant \varepsilon n$, we get $e_H(S,T) > 0$.*

Lemma 4 ([6, Lemma 3.5])**.** *For every integer $k \geqslant 1$ and every $\varepsilon > 0$ there exists an integer a such that the following holds. Let H be a graph on at least am vertices such that for every pair of disjoint sets S, $T \subseteq V(H)$ with $|S|$, $|T| \geqslant \varepsilon m$ we have $e_H(S,T) > 0$. Then the following holds:*

(P2$_m$) *For every family $A_1, \ldots, A_{k+1} \subseteq V(H)$ of pairwise disjoint sets each of size at least εam, there is a path $P_m = (x_1, \ldots, x_m)$ in H with $x_i \in A_j$ for all $1 \leqslant i \leqslant m$, where $j \equiv i \pmod{k+1}$.*

Note that the hypothesis on H in Lemma 4 is (P1$_m$) from Lemma 3. Therefore, roughly speaking, Lemma 4 tells us that (P1$_m$) implies (P2$_m$).

§3. (2, 3)-graphs

In this section we introduce a structure that helps us to transfer some ideas from the graph case to the hypergraphs setting. A $(2,3)$-*graph* $F = (V, E)$ consists of a set of vertices V and a set E of 2-*edges* of the form uv and 3-*edges* of the form $uv(w)$, for distinct vertices $u, v, w \in V$.[1] A sequence of vertices $P = (x_1, \ldots, x_m)$ is a $(2,3)$-path of length m in F if for every $i = 1, \ldots, m-1$ either $x_i x_{i+1} \in E$ or $x_i x_{i+1}(w_i) \in E$ for some $w_i \in V \smallsetminus \{x_1, \ldots, x_m\}$, with all the w_i distinct.

Given pairwise disjoint sets $V_1, \ldots, V_{k+1}$, we say that an edge $uv \in E(F)$ ($uv(w) \in E(F)$) is a *transversal with respect to* $V_1, \ldots, V_{k+1}$, if u and v (u, v, and w) are in different sets V_i. When the sets $V_1, \ldots, V_{k+1}$ are clear from the context we say that the edge is a *transversal*.

We want to prove that if a sufficiently large $(2,3)$-graph $F = (V, E)$ contains no $(2,3)$-path with n vertices, then there exist large disjoint sets $V_1, \ldots, V_k \subseteq V$ such that E contains no transversals and that there is no edge $uv(w)$ with $u \in V_1 \cup \cdots \cup V_{k-1}$ and $v, w \in V_k$. The last property is only required to support our inductive proof of Theorem 1. To prove this we use a *Depth First Search (DFS)* algorithm. For example, Ben-Eliezer, Krivelevich, and Sudakov [3] used a DFS algorithm to find long paths in expanding graphs to obtain bounds on the size-Ramsey number of directed paths. Their algorithm traverses the vertices of the input graph and maintains a set S of vertices that are fully dealt with, a set U of currently active vertices, and a set J of vertices that were not considered so far. The set U always spans a path and in every step this path is either extended by adding a vertex from J to it or this is not possible, and then its last vertex is removed and is moved to S. It follows that there cannot be any edges between S and J.

Algorithm 1 below is the algorithm just described adjusted to the setting of $(2,3)$-graphs. The input of the algorithm is a $(2,3)$-graph $F = (V, E)$, disjoint subsets of vertices $V_1, \ldots, V_k$, and an ordering of the vertices $V = \{v_1, \ldots, v_N\}$. During the algorithm we maintain sets S, T, W_S, W_U, T_i for $i \in [k]$ and a $(2,3)$-path U as follows:

- $S \subseteq V'$ is the set of vertices that are fully dealt with.
- $W_S \subseteq V$ is the set of vertices w that were 'used' by vertices from S.
- U contains the currently active vertices in a $(2,3)$-path.
- $W_U \subseteq V$ is the set of vertices w that are 'used' by the path U.
- $T_1 \cup \cdots \cup T_k$ are disjoint and $T_i \subset V_i$ for $i \in [k]$.

In every step of the algorithm, either the $(2,3)$-path U is extended by adding a vertex from T_k to it or this is not possible, and the last vertex from U is removed and put into S. While the algorithm runs, after each execution of the while loop, we have the following invariants, where m is the length of the $(2,3)$-path U:

(A1) $U = (u_1, \ldots, u_m)$ is a $(2,3)$-path and W_U is the set of the vertices w in the edges $u_i u_{i+1}(w)$ $(1 \leqslant i < m)$ in the $(2,3)$-path U.

(A2) S, $U \subseteq V_k$, $W_U \subseteq T_1 \cup \cdots \cup T_{k-1}$, $T_i \subseteq V_i$ for $i \in [k]$, $|W_S| \leqslant |S|$, and $|W_U| \leqslant \max\{0, m-1\}$.

This process is described in Algorithm 1.

Lemma 5. *Algorithm 1 terminates and Properties (A1) and (A2) hold throughout.*

[1]To be precise, uv is shorthand for $\{u, v\}$ and $uv(w)$ is shorthand for $(\{u, v\}, w)$.

Algorithm 1: DFS algorithm for traversing a $(2,3)$-graph.

Input : A $(2,3)$-graph $F = (V, E)$, disjoint subsets of vertices $V_1, \dots, V_k$, and an ordering of the vertices $V = \{v_1, \dots, v_N\}$.

1 Define $m \leftarrow 0$, $S \leftarrow \varnothing$, $W_S \leftarrow \varnothing$, $W_U \leftarrow \varnothing$, $T_i \leftarrow V_i$ for $1 \leqslant i \leqslant k$;
2 **while** $T_k \neq \varnothing$ **do**
3   **if** $m = 0$ **then**
4   Let v be the vertex with smallest index from T_k;
5         $u_1 \leftarrow v$, $m \leftarrow 1$, $T_k \leftarrow T_k \setminus \{v\}$;
6   **else**
7   Let $T_{\text{ext}} \leftarrow \{v \in T_k \colon u_m v \in E \text{ or } u_m v(w) \in E \text{ with } w \in T_1 \cup \dots \cup T_k\}$;
8   **if** $T_{\text{ext}} \neq \varnothing$ **then**
9   Let v be the vertex with the smallest index from T_{ext};
10             $u_{m+1} \leftarrow v$, $T_k \leftarrow T_k \setminus \{v\}$;
11   **if** $u_m u_{m+1} \notin E$ **then**
12   Let $w \in T_1 \cup \dots \cup T_k$ be the vertex of smallest index such that $u_m u_{m+1}(w) \in E$; `// There is one because` $u_{m+1} \in T_{\text{ext}}$.
13                 $W_U \leftarrow W_U \cup \{w\}$ and $T_i \leftarrow T_i \setminus \{w\}$, where $w \in T_i$;
14             $m \leftarrow m + 1$;
15   **else**
16             $S \leftarrow S \cup \{u_m\}$;
17   **if** $u_{m-1} u_m \notin E$ **then**
18   Let $u_{m-1} u_m(w) \in E$ with $w \in W_U$ be the edge used by the $(2,3)$-path; `// This is well defined by (A1).`
19                 $W_S \leftarrow W_S \cup \{w\}$ and $W_U \leftarrow W_U \setminus \{w\}$;
20             $m \leftarrow m - 1$;

Proof of Lemma 5. Observe that (A1) and (A2) hold when we initialise the sets and put $m = 0$ on line 1. Assume that we are in some step of the algorithm, where (A1) and (A2) hold and we have vertices $U = (u_1, \dots, u_m)$ forming a $(2,3)$-path (this is true because of (A1)).

Now we consider the next execution of the while loop. We know from (A1) that W_U contains exactly the vertices used in the edges $u_i u_{i+1}(w)$ for $i = 1, \dots, m-1$. Either we extend the path by an edge $u_m v$ or $u_m v(w)$ (lines 7 and 10) where w is added to W_U if needed (line 13), or we remove an edge $u_{m-1} u_m$ or $u_{m-1} u_m(w)$ (lines 16 and 19) where w is removed from W_U if needed (line 19). Therefore, (A1) still holds.

For (A2) it is easy to see that $T_i \subseteq V_i$ for $i \in [k]$, as in the beginning of the execution we have $T_i = V_i$ and no vertex is added to T_i. Also, since every w in W_U comes from $T_1 \cup \dots \cup T_k$, we have $W_U \subseteq T_1 \cup \dots \cup T_{k-1}$ (see lines 7 and 19). Since every vertex of S comes from U (line 16) and every vertex of U comes from $T_k \subset V_k$ (lines 5, 7 and 10), which implies that S, $U \subset V_k$. To prove that $|W_S| \leqslant |S|$, it is enough to observe that line 19 can only be executed after an execution of line 16. Similarly, we have $|W_U| \leqslant m - 1$ with $m \geqslant 2$, because line 13 can only be executed after an execution of line 10, and $|W_U| = 0$ with $m = 1$, because on line 5 nothing is added to W_U. Thus, (A2) also remains true.

It remains to show that the algorithm terminates. In every execution of the while loop, either one vertex from $T_k \subseteq V_k$ is added to the path U (lines 5 and 10) or moved from U to S (line 16). Therefore, after at most $2|V_k|$ steps we have $T = \varnothing$, and the algorithm terminates. □

We are ready to prove the aforementioned result on $(2,3)$-graphs F with no long $(2,3)$-paths.

Lemma 6. *Let k, c and n be positive integers and let $F = (V, E)$ be a $(2,3)$-graph on at least $5^{k-1}cn$ vertices. If F contains no $(2,3)$-path with n vertices, then there exist disjoint sets $V_1, \dots, V_k \subseteq V$ of size at least cn such that no edge from E is a transversal and there is no edge $uv(w)$ with $u \in V_1 \cup \dots \cup V_{k-1}$ and $v, w \in V_k$.*

Proof. We prove the result by induction on k. For $k = 1$ the result follows by putting $V_1 = V(F)$. Thus let $k \geqslant 1$ and assume the statement holds for k.

To prove the result for $k + 1$, let F be a $(2,3)$-graph on $5^k cn$ vertices which does not have a path of length n. In particular, F does not have a path of length $5n$, and by the assumption on k, there exist disjoint sets $V_1, \cdots, V_k$, each of size $5cn$, such that no edge is a transversal, and there is no edge $uv(w)$ with $u \in V_1 \cup \dots \cup V_{k-1}$ and $v, w \in V_k$. We run Algorithm 1 with input F, k and $V_1, \dots, V_k$.

First, we prove that at any point in the execution of the algorithm, no edge is a transversal with respect to $T_1 \cup \dots \cup T_k, S$. Suppose for a contradiction that at some point there is an edge which is a transversal. Note that $T_k \subseteq V_k$ and $S \subseteq V_k$. If uv is this edge, then by the induction hypothesis and without loss of generality we have $u \in S$ and $v \in T_k$. This implies that when u was moved from U to S (line 16), the set U could have been extended, which means that $T_{\text{ext}} \neq \varnothing$ and line 16 would not have been executed, a contradiction. Now, assume $uv(w)$ is the transversal. Since $T_i \subseteq V_i$ for $i \in [k]$ and $S \subseteq V_k$, we have $w \in T_1 \cup \dots \cup T_{k-1}$. Again, by the induction hypothesis and without loss of generality we have $u \in S$ and $v \in T_k$. Similarly as when we have an edge uv, at the time u was moved from U to S, the set U could have been extended, a contradiction.

Now we prove that at any point in the execution of the algorithm, there is no edge $uv(w)$ with $u \in T_1, \dots, T_{k-1}, S$ and $v, w \in T_k$. Suppose for a contradiction that at some point there is such edge $uv(w)$. By the induction hypothesis we have $u \in S$ and $v, w \in T_k$, which again gives a contradiction as U could have been extended.

Note that since F has at least $5^{k-1}cn$ vertices and no $(2,3)$-path with n vertices, we have $|S| = cn$ at some point of the execution of Algorithm 1. Let U, W_U, S, W_S and $T_1, \dots, T_k$ be the sets at that moment. Note that $|W_S| \leqslant |S| = cn$ and, since there is no n-vertex $(2,3)$-path in F, we have $|U|$, $|W_U| \leqslant n$. Therefore, $|T_i| \geqslant |V_i| - 2cn \geqslant cn$ for $i \in [k-1]$ and $|T_k| \geqslant |V_k| - 4cn \geqslant cn$. Put $V'_i = T_i$ for $i \in [k-1]$, $V'_k = S$ and $V'_{k+1} = T_k$. The sets $V'_1, \dots, V'_{k+1}$ satisfies the requirements of the lemma. □

§4. Proof of Theorem 1

In this section we prove our main theorem. We first define the following constants:

$$\ell := 17, \quad k := 2\ell, \quad \varepsilon := 1/(k+1), \quad t := 8k + 40k^2 + 5, \quad \text{and} \quad t' := r_2(K_t^{(3)}),$$

where $r_s(K_t^{(3)}) := \min\{n : K_n^{(3)} \to (K_t^{(3)})_s\}$ is the classical Ramsey-number for hypergraphs. We start by obtaining a graph G with bounded maximum degree and some nice pseudorandom properties. Let $a_{\mathrm{L}.4}$ be large enough to apply Lemma 4 with k and ε and set

$$c := \varepsilon a_{\mathrm{L}.4}\ell.$$

Let $a_{\mathrm{L}.6} = 5^k$ and note that $a_{\mathrm{L}.6}$ is large enough to apply Lemma 6 with $k+1$, c and n and set

$$a := 2a_{\mathrm{L}.6}.$$

Lemma 3 applied with ε and a provides a constant b. Let n be sufficiently large. Then, from Lemma 3 we know that there is a graph G on an vertices with maximum degree b such that (P1$_n$) holds. Fix such a graph G.

Now let $G^k(t')$ be the graph obtained from G^k – the k-th power of G – by replacing every vertex by a $K_{t'}$ and every edge by a $K_{t',t'}$. Finally, H is the 3-uniform hypergraph with vertex set $V(G^k(t'))$ and a triple of vertices xyz is an edge in H if and only if xyz forms a triangle in $G^k(t')$. For every $v \in V(G)$ we denote by $H(v)$ the corresponding cluster consisting of a $K_{t'}^{(3)}$ in H. We claim that $H \to (P_n^{(3)})_2$. Since $|V(H)| = at'n$ and $\Delta(H) \leqslant b^{2k+2}t'^3$, this would prove Theorem 1.

The rest of the proof is devoted to proving that $H \to (P_n^{(3)})_2$. Fix a 2-colouring of the triples of H. As $t' \geqslant r_2(K_t^{(3)})$ for every $v \in V(G)$ the cluster $H(v)$ either contains a red or blue copy of $K_t^{(3)}$. W.l.o.g. there is a set of vertices $V \subseteq V(G)$ with $|V| \geqslant an/2 = a_{\mathrm{L}.6}n$ such that for all $v \in V$ the cluster $H(v)$ contains a blue copy of $K_t^{(3)}$, which we denote by $H'(v)$. We let $H' \subseteq H$ be the 3-graph induced by the clusters $H'(v)$ for $v \in V$.

We will define an auxiliary $(2,3)$-graph F on the vertex set V, whose edges will indicate that we can walk between the clusters using blue triples of H'. Formally, for $u, v \in V$ a $(2,2)$-*connector* between the clusters $H'(u)$ and $H'(v)$ consists of four vertices $x_1, x_2 \in H'(u)$ and $y_1, y_2 \in H'(v)$ such that $x_1x_2y_1$ and $y_1y_2x_1$ are triples of H. Similarly, for $u, v, w \in V$ a $(2,1,2)$-*connector* between the clusters $H'(u)$ and $H'(v)$ through $H'(w)$ consists of five vertices $x_1, x_2 \in H'(u)$, $z \in H'(w)$, and $y_1, y_2 \in H'(v)$ such that x_1x_2z, x_1zy_1, and zy_1y_2 are triples of H; see Figure 1. We then define a $(6,6)$-*connector* ($(6,3,6)$-*connector*) between $H'(u)$ and $H'(v)$ (through $H'(w)$) as the disjoint union of three $(2,2)$-connectors ($(2,1,2)$-connectors) between $H'(u)$ and $H'(v)$ (through $H'(w)$).

Let F be a $(2,3)$-graph on the vertex set V with the following two types of edges:

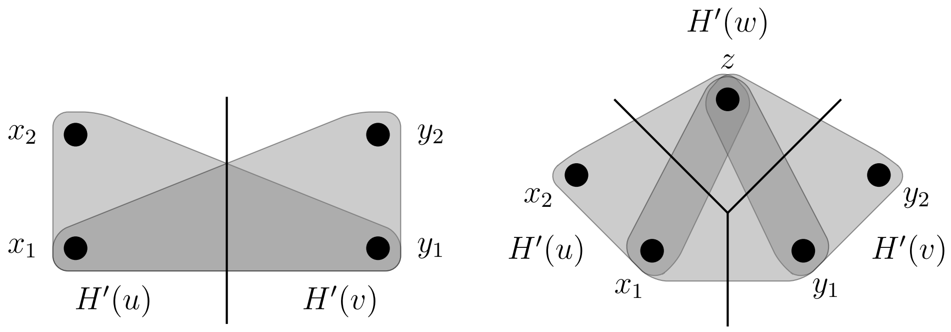

FIGURE 1. A $(2,2)$-connector and a $(2,1,2)$-connector.

(*i*) $uv \in E(F)$ if and only if there is a $(6,6)$-connector in blue between the corresponding clusters $H'(u)$ and $H'(v)$;

(*ii*) $uv(w) \in E(F)$ if and only if there is a $(6,3,6)$-connector in blue between the corresponding clusters $H'(u)$ and $H'(v)$ through $H'(w)$.

Suppose that F contains a $(2,3)$-path on n vertices. We can turn this $(2,3)$-path into a blue tight path $P_n^{(3)}$ in H as follows: we choose a $(2,2)$-connector or a $(2,1,2)$-connector for every edge of the $(2,3)$-path in such a way that they are all pairwise vertex-disjoint. This is possible, because we have $(6,6)$-connectors and $(6,3,6)$-connectors available. Then we simply follow the $(2,3)$-path, using the blue triples inside the clusters to connect the $(2,2)$-connectors and $(2,1,2)$-connectors. Thus, in this case, we are able to obtain a blue $P_n^{(3)}$ in H and we are done.

We assume that F contains no $(2,3)$-path on n vertices. From Lemma 6, there exist pairwise disjoint sets $V_1, \dots, V_{k+1} \subseteq V$ of size at least cn such that no edge from $E(F)$ is a transversal. We may assume that all these sets V_j have size exactly cn. Let $G' = G[V_1 \cup \dots \cup V_{k+1}]$ and set $m := \ell n$.

We now want to find a path P_m alternating through $V_1, \dots, V_{k+1}$ with edges in $G' \subseteq G$ using Lemma 4. Since $c = \varepsilon a_{\text{L.4}} \ell$ and $\varepsilon = 1/(k+1)$, we have $|V(G')| = (k+1)cn = a_{\text{L.4}} \ell n = a_{\text{L.4}} m$. Also, we have $|V_i| = cn = \varepsilon a_{\text{L.4}} m$ for $1 \leqslant i \leqslant k+1$. As G' is an induced subgraph of G and property (P1$_n$) holds in G, property (P1$_m$) does hold for G'. Therefore, by Lemma 4, we conclude that there is a path $P_m = P_{\ell n}$ with vertices alternating through $V_1, \dots, V_{k+1}$ and with edges in $G' \subseteq G$.

This path $P_{\ell n}$ gives us the kth power $P_{\ell n}^k$ in G^k. By the choice of $V_1, \dots, V_{k+1}$ no edge of $P_{\ell n}^k$ is from $E(F)$ and also no triangle in $P_{\ell n}^k$ induces an edge $uv(w) \in E(F)$. It remains to turn this $P_{\ell n}^k$ into a red $P_n^{(3)}$ in H'.

Claim 7. *If there is a $P_{\ell n}^k$ in G^k that does not contain any edges from F, then there is a red $P_n^{(3)}$ in H'.*

Let $P_{\ell n}^k = (v_1, \dots, v_{\ell n})$ and recall $H'(v_i)$ is the cluster in H' corresponding to the vertex v_i for $i = 1, \dots, \ell n$. We want to remove all vertices of H' which belong to blue $(2,2)$-connectors and $(2,1,2)$-connectors from clusters along edges and triangles of $P_{\ell n}^k$.

In $P_{\ell n}^k$ every vertex v_i is incident to at most $2k$ other vertices in $\{v_1, \dots, v_{\ell n}\}$ ($2k$ is the maximum degree of the v_i in $P_{\ell n}^k$). Also, every v_i is contained in at most $4k^2$ triangles of $P_{\ell n}^k$ together with two other vertices in $\{v_1, \dots, v_{\ell n}\}$.

Let u and v be neighbours in $P_{\ell n}^k$. Since there is no blue $(6,6)$-connector between $H'(u)$ and $H'(v)$, there are at most two $(2,2)$-connectors that do not overlap between $H'(u)$ and $H'(v)$, which can both be deleted by removing at most 4 vertices in each cluster. Let u, v, and w be vertices that form a triangle in $P_{\ell n}^k$. Since there is no blue $(6,3,6)$-connector between $H'(u)$, $H'(v)$, and $H'(w)$, there are at most six $(2,1,2)$-connectors that do not overlap, two for each possibility to place the single vertex. These can be deleted by removing at most 10 vertices from each cluster.

By the above argument, we have to delete at most $4(2k) + 10(4k^2) \leqslant t - 5$ vertices from every cluster to get rid of all $(2,2)$-connectors and $(2,1,2)$-connectors. Let $H^*(v_i) \subseteq H'(v_i)$ be the remainder of the corresponding cluster in H', and note that $|H^*(v_i)| \geqslant 5$ for $i = 1, \dots, \ell n$.

A tuple (u, v) is an *end-tuple* of a tight path with at least 4 vertices if u and v are consecutive vertices in the path and u is contained in exactly two edges and v is contained in exactly one. The two tuples (u, v) and (v, w) are the end-tuples of the tight path (u, v, w) of length 3. Furthermore, every tuple (u, v) is an end-tuple of the tight path (u, v) of length 2.

Definition 8. *For $i = 1, \dots, n-1$ we say that the quadruple (u_1, u_2, w_1, w_2) satisfies property Q_i if the following conditions hold:*

(1) u_1, u_2, w_1, w_2 are distinct vertices from H' such that the pairs u_1, u_2 and w_1, w_2 are in clusters $H_r, H_s \in \{H^*(v_{(i-1)\ell+1}), \dots, H^*(v_{i\ell})\}$, respectively, where $r \neq s$;

(2) each of (u_1, u_2) and (w_1, w_2) is an end-tuple of a red tight path of length at least $i+1$ with vertices in $H^*(v_1) \cup \dots \cup H^*(v_{i\ell})$.

To prove Claim 7, i.e., construct $P_n^{(3)}$ in red, it is then sufficient to construct a quadruple satisfying property Q_{n-1}. We will construct this quadruple inductively. The base case Q_1 asks for two paths of length 2 and, therefore, it is enough to choose any pair u_1, u_2 from $H^*(v_1)$ and w_1, w_2 from $H^*(v_2)$. Therefore, the following is immediate:

$$\text{There exists a quadruple } (u_1, u_2, w_1, w_2) \text{ for which } Q_1 \text{ holds.} \tag{1}$$

We will inductively find a quadruple (u_1, u_2, w_1, w_2) satisfying property Q_i for every $i = 2, \dots, n-1$. Ultimately, after n steps, this gives us $P_n^{(3)}$ in red in H'. Suppose (u_1, u_2, w_1, w_2) satisfies property Q_{i-1} for some $1 < i \leqslant n-1$. In the inductive step, we obtain (x_1, x_2, y_1, y_2) satisfying property Q_i by extending one of the paths ending in (u_1, u_2) or (w_1, w_2) to get two longer paths ending in (x_1, x_2) and (y_1, y_2). This mainly relies on the absence of $(2,2)$-connectors and $(2,1,2)$-connectors in blue and that there are ℓ clusters $H^*(v_{(i-1)\ell+1}), \dots, H^*(v_{i\ell})$ to choose x_1, x_2, y_1, y_2 from. As $k \geqslant 2\ell$, all edges are present between these clusters and the clusters H_r and H_s containing u_1, u_2 and w_1, w_2, respectively.

Fact 9. *Let $1 < i \leqslant n-1$ and suppose (u_1, u_2, w_1, w_2) satisfies property Q_{i-1}. Then there is a quadruple (x_1, x_2, y_1, y_2) that satisfies property Q_i.*

Proof. We will find (x_1, x_2, y_1, y_2) following the strategy sketched above. Since there are no blue $(2,2)$-connectors between the clusters corresponding to u_1, u_2 and w_1, w_2, and all possible triples between these clusters are edges in H', then either the triple $u_1u_2w_2$ or the triple $w_1w_2u_2$ is red, say, w.l.o.g., $u_1u_2w_2$ is red. We let the red path of length at least i that ends in (u_1, u_2) be called P_{red} and note that the triple $u_1u_2w_2$ already extends this path. We will show that it is possible to further extend this path to obtain two longer red tight paths with ends (x_1, x_2) and (y_1, y_2), respectively, such that (x_1, x_2, y_1, y_2) satisfies property Q_i.

Notice that, as $\ell \geqslant 17$, by the pigeonhole principle there are nine sets $X_1, \dots, X_9$ each contained in a different cluster from $H^*(v_{(i-1)\ell+1}), \dots, H^*(v_{i\ell})$ and of size $|X_j| \geqslant 3$ for $j \in [9]$ (here we use that $|H^*(v_i)| \geqslant 5$) such that either

$$\text{for all } j \in [9] \text{ and every } x \in X_j \text{ the triple } u_2w_2x \text{ is red} \tag{2}$$

or

$$\text{for all } j \in [9] \text{ and every } x \in X_j \text{ the triple } u_2w_2x \text{ is blue.} \tag{3}$$

We first consider the case where (2) holds. It is enough to assume that we have $X_1, \dots, X_5$ and $|X_j| \geqslant 2$ for $j \in [5]$. If there are two sets X, Y from $X_1, \dots, X_5$ and $x_1, x_2 \in X$, $y_1, y_2 \in Y$ such that the triples $w_2x_1x_2$ and $w_2y_1y_2$ are red, then the quadruple (x_1, x_2, y_1, y_2) satisfies property Q_i as we can obtain two longer red paths by extending P_{red} following $u_1u_2w_2x_1x_2$ and $u_1u_2w_2y_1y_2$, respectively. Otherwise, there are two sets X, Y and $x_1, x_2 \in X$, $y_1, y_2 \in Y$ such that the triples $w_2x_1x_2$ and $w_2y_1y_2$ are blue. As there is no blue $(2,1,2)$-connector, the triple $w_2x_1y_1$ is red. There is no blue $(2,2)$-connector between the corresponding clusters, so either the triple $x_1y_1y_2$ or $y_1x_1x_2$ is red, say w.l.o.g. $x_1y_1y_2$ is red. This extends P_{red} by following $u_1u_2w_2x_1y_1y_2$ and gives the end-tuple (y_1, y_2). Repeating the same argument, which is possible, because there were five sets available (sets $X_1, \dots, X_5$), we get an end-tuple (z_1, z_2) that extends P_{red} to a longer red path and, thus, a quadruple (y_1, y_2, z_1, z_2) satisfying Q_i.

In the case where (3) holds we proceed as follows. As for all $j \in [9]$ there is no blue $(2,1,2)$-connector between the clusters of u_1, u_2 and w_1, w_2 and X_j, we have for every $x \in X_j$ that either the triple w_1w_2x or the triple u_1u_2x is red. Then we can assume by the pigeonhole principle w.l.o.g. (we will not use the triple $u_1u_2w_1$) that there are sets $X'_j \subseteq X_j$ with $|X'_j| \geqslant 2$ for $j \in [5]$ (here we use that $|X_j| \geqslant 3$) such that

$$\text{for all } j \in [5] \text{ and every } x \in X'_j \text{ the triple } w_1w_2x \text{ is red.}$$

Now we can continue exactly as in the case where (2) holds, with u_1u_2 replaced by w_1w_2 throughout and extending the path with end-tuple (w_1, w_2). Observing that the red tight paths that we have constructed have length at least $i+1$, we see that Fact 9 is proved. □

Fact 9 together with (1) finishes the proof of Claim 7 and hence the proof of Theorem 1 is complete.

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