Large monochromatic triple stars in edge colourings

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Abstract

Following problems posed by Gyárfás, we show that for every r-edge-colouring of K_n there is a monochromatic triple star of order at least $n/(r-1)$, improving a results by Ruszinkó's.

An edge colouring of a graph is called a local r-colouring if every vertex spans edges of at most r distinct colours. We prove the existence of a monochromatic triple star with at least $rn/(r^2-r+1)$ vertices in every local r-colouring of K_n.

1 Introduction

A very simple observation, remarked by Erdős and Rado, is that when the edges of K_n are 2-coloured there exists a monochromatic spanning component. One can generalize this and look for large monochromatic components satisfying certain conditions. For example, it is an easy exercise to show that every 2-colouring of K_n has a spanning component of diameter at most 3 (see [1], [2]). As a further generalization, one can consider edge colourings with more than two colours. Gyárfás [6] extended the above observation by showing that every r-colouring of K_n has a monochromatic component with at least $n/(r-1)$ vertices. This is tight when there exists an affine space of order $r-1$ and $(r-1)^2$ divides n. Füredi [4] improved this bound in the case when there exists no affine space of order $r-1$, showing that for such r every r-colouring of K_n has a monochromatic component with at least $n/(r-1-(r-1)^{-1})$ vertices.

A *double star* is a tree obtained by joining the centres of two stars by an edge. Gyárfás [5] proposed the following problem.

Problem 1. *Is it true that for $r \geq 3$ every r-colouring of K_n contains a monochromatic double star of size at least $n/(r-1)$?*

For $r=2$ the answer to this question is negative. It is shown in [8] that when K_n is 2-coloured there is a monochromatic double star of size at least $3n/4$. This can be shown to be asymptotically tight using random graphs. The best known result so far for $r \geq 3$ was obtained by Gyárfás and Sárközy [8]. They showed that when the edges of K_n are r-coloured there is a monochromatic double star of size at least $\frac{n(r+1)+r-1}{r^2}$.

A weaker version of the above problem is as follows.

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Problem 2. *Is there a constant d for which in every r-colouring of K_n there exists a monochromatic component of diameter at most d and size at least $n/(r-1)$?*

Note that an affirmative answer to the first problem implies an affirmative answer to this one with $d = 3$, which would be best possible (see [3]). Ruszinkó [10] solved the last problem with $d = 5$, showing that for every r-colouring of K_n there is a monochromatic component of diameter at most 5 with at least $n/(r-1)$ vertices.

The first main result of this short note proves a weaker version of the first problem. A *triple star* is a tree obtained by joining the centres of three stars by a path of length 2.

Theorem 1. *Let $G = K_n$ be r-edge-coloured with $r \geq 3$. Then G contains a monochromatic triple star with at least $n/(r-1)$ vertices.*

Note that this is sharp in certain cases, namely whenever $n/(r-1)$ is a sharp lower bound for general monochromatic components in r-colourings of K_n. We remark that the assertion of Theorem 1 does not holds for $r = 2$. Indeed, consider a random red and blue colouring of K_n. With high probability, for any three vertices, the union of their red neighbourhoods has size $(1 + o(1))\frac{7n}{8}$ and similarly for the union of their blue neighbourhoods. In particular, the size of the largest monochromatic triple star is $(1 + o(1))\frac{7n}{8}$.

As an immediate corollary of Theorem 1 we answer problem 2 with $d = 4$, improving Ruszinkó's result.

Corollary 2. *Let $r \geq 3$. In every r-colouring of K_n there is a monochromatic subgraph of diameter at most 4 on at least $n/(r-1)$ vertices.*

A *local r-colouring* is an edge colouring in which for every vertex the edges incident to it have at most r distinct colours. In [7] it is shown that in every local r-colouring of K_n there is a monochromatic component with at least $\frac{rn}{r^2-r+1}$ vertices. This is sharp when there exists a projective plane of order $r-1$ and $r^2 - r + 1$ divides n. In [8] it is shown that in local r-colourings of K_n there is a monochromatic double star of size at least $\frac{(r+1)n+r-1}{r^2+1}$. Moreover, it is shown that for local 2-colouring of K_n there exists a monochromatic double star of size at least $2n/3$, and, as mentioned above, this is a sharp lower bound for the size of general monochromatic connected components. Our second main result shows that the above lower bounds for monochromatic components can be achieved also for components which are triple stars.

Theorem 3. *Let $G = K_n$ be r-locally-coloured with $r \geq 3$. Then G contains a monochromatic triple star with at least $\frac{rn}{r^2-r+1}$ vertices.*

As before, the following corollary is immediate.

Corollary 4. *Let $r \geq 3$. In every r-local-colouring of K_n there exists a monochromatic component of diameter at most 4 with at least $\frac{rn}{r^2-r+1}$ vertices.*

We prove Theorem 1 in Section 2, and Theorem 3 in Section 3. In the last section 4 we finish with some concluding remarks and open problems.

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of Theorem 1. We assume to the contrary of the statement in the theorem that G contains no monochromatic triple star of the given size. Let G_1 be a subgraph of G which is a monochromatic double star of maximal order and let U be its vertex set. Denote the colour of the edges of G_1 by r. By our assumption $|U| < n/(r-1)$. Let $a > 0$ satisfy $|U| = n/(r-1) - a$ (note that a may not be an integer).

Consider the bipartite graph G_2 with bipartition $U \cup (V(G) \setminus U)$ and edge set E, containing the edges between U and $V(G) \setminus U$ not coloured by r in G. Note that for every vertex $u \in U$ less than a edges between u and $V(G) \setminus U$ have colour r, as otherwise there would be an r-coloured triple star with at least $n/(r-1)$ vertices, contradicting our assumption. Therefore

$$|E| > |U|\big(n - |U|\big) - a|U|. \tag{1}$$

We use the following lemma which is due to Mubayi [2] and Liu, Morris and Prince [9]. We present the proof here for the sake of completeness.

Lemma 5. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$. Then G contains a double star with at least $\big(\frac{1}{|A|} + \frac{1}{|B|}\big)|E|$ vertices.*

Proof. For a vertex $v \in V$, let $d(v)$ denote the degree of v in G and for an edge $e = (a, b) \in E$, let $c(e) = d(a) + d(b)$. By the Cauchy-Schwartz inequality,

$$\sum_{e\in E} c(e) = \sum_{a\in A} d(a)^2 + \sum_{b\in B} d(b)^2 \geq \frac{1}{|A|}\Big(\sum_{a\in A} d(a)\Big)^2 + \frac{1}{|B|}\Big(\sum_{b\in B} d(b)\Big)^2 = \Big(\frac{1}{|A|} + \frac{1}{|B|}\Big)|E|^2.$$

Therefore, there is an edge $e \in E$ with $c(e) \geq \big(\frac{1}{|A|} + \frac{1}{|B|}\big)|E|$, i.e. G contains a double star of the required order. $\square$

By considering the edges with the majority colour, the lemma implies that G_2 has a monochromatic double star G_3 with at least $\big(\frac{1}{|U|} + \frac{1}{n-|U|}\big)\frac{|E|}{r-1}$ vertices. Using inequality (1) for the size of E and the expression for the size of U, G_3 has at least the following number of vertices.

$$\begin{aligned}
&\frac{1}{r-1} \cdot \frac{n}{|U|(n-|U|)} \cdot \big(\,|U|(n-|U|) - a|U|\,\big) = \\
&\frac{n}{r-1} - a\frac{n}{r-1}\Big(\frac{1}{\frac{r-2}{r-1}n + a}\Big) > \\
&\frac{n}{r-1} - \frac{a}{r-2} \geq \frac{n}{r-1} - a = |U|
\end{aligned}$$

Note that we use here the fact that $r \geq 3$. This implies that G_3 has more than $|U|$ vertices, contradicting the choice of U as the vertex set of the largest monochromatic double star of G. We have thus reached a contradiction to the initial assumption, i.e. G contains a triple star of the required size. $\square$

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of Theorem 3. As in the proof of Theorem 1, we take U to be the vertex set of the largest monochromatic double star, and assume it has $\frac{rn}{r^2-r+1} - a$ vertices, where $a > 0$. We define the bipartite graph G_2 as before, and obtain the same inequality (1) for $|E|$. The following lemma generalizes Lemma 5 from the previous section. A weaker form of this lemma appears in [7].

Lemma 6. *Let a bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$ be edge coloured. Let r, t be such that every vertex $x \in A$ is incident to edges of at most r distinct colours, and every vertex $y \in B$ is incident to edges of at most t colours. Then G contains a monochromatic double star with at least $\left(\frac{1}{|A|r} + \frac{1}{|B|t}\right)|E|$ vertices.*

Proof. For a vertex $v \in A \cup B$ denote by $I(v)$ the number of colours used in the set of edges in G incident with v. For a colour k denote by $d_k(v)$ the number of k-coloured-edges containing v. For en edge $e = (a, b)$ in G of colour k let $c(e) = d_k(a) + d_k(b)$. Then by the Cauchy-Schwartz inequality, using the properties of the colouring,

$$\sum_{e\in E} c(e) = \sum_{a\in A}\sum_{k\in I(a)} d_k(a)^2 + \sum_{b\in B}\sum_{k\in I(b)} d_k(b)^2 \geq$$
$$\frac{1}{|A|r}\Big(\sum_{a\in A}\sum_{k\in I(a)} d_k(a)\Big)^2 + \frac{1}{|B|t}\Big(\sum_{b\in B}\sum_{k\in I(b)} d_k(b)\Big)^2 = \Big(\frac{1}{|A|r} + \frac{1}{|B|t}\Big)|E|^2.$$

It follows that G contains a monochromatic double star of the required order. □

Note that every vertex in $V(G) \setminus U$ spans edges of at most r colours in G_2 and every vertex in U spans edges with at most $r-1$ colours in G_2, using the fact that G is locally r-coloured, and the definition of G_2. Thus G contains a monochromatic double star with at least the following number of vertices.

$$\Big(\frac{1}{|U|(r-1)} + \frac{1}{(n-|U|)r}\Big)|E| > \Big(\frac{1}{|U|(r-1)} + \frac{1}{(n-|U|)r}\big)(|U|(n-|U|) - a|U|\Big) =$$
$$\frac{n-|U|}{r-1} + \frac{|U|}{r} - \frac{a}{r-1} - a\frac{|U|}{(n-|U|)r} =$$
$$\frac{(r-1)n}{r^2-r+1} + \frac{a}{r-1} + \frac{n}{r^2-r+1} - \frac{a}{r} - \frac{a}{r-1} - a\cdot\frac{\frac{rn}{r^2-r+1} - a}{\frac{(r-1)^2rn}{r^2-r+1} + a} \geq$$
$$\frac{rn}{r^2-r+1} - \frac{a}{r} - \frac{a}{(r-1)^2} > \frac{rn}{r^2-r+1} - a = |U|.$$

As in Theorem 1, we reached a contradiction to the choice of U, thus we have a monochromatic triple star of the required size. □

4 Concluding Remarks

Problem 1, which is the original question posed by Gyárfás, remains open. Is it true that for $r \geq 3$ every r-colouring of K_n contains a monochromatic triple star with at least $n/(r-1)$ vertices? It may also be interesting to consider the weaker version of this question, taking $d = 3$ in problem

2. Does every r-colouring of K_n contain a diameter 3 monochromatic subgraph of size at least $n/(r-1)$? Finally, it may be interesting to address the same questions in the context of local r-colourings (for $r \geq 3$). Namely, is it true that every local r-colouring contains a component of diameter at most 3 with at least $\frac{rn}{r^2-r+1}$ vertices? If so, is there such a component which is a double star?

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