Ascending subgraph decompositions

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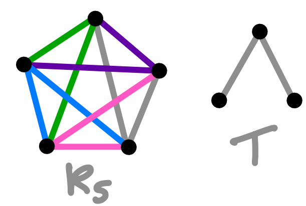

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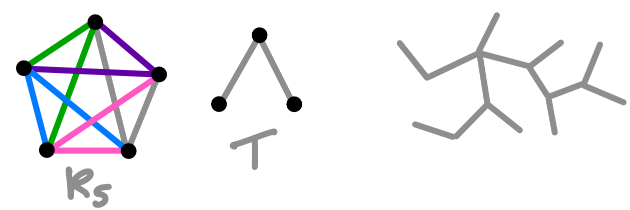

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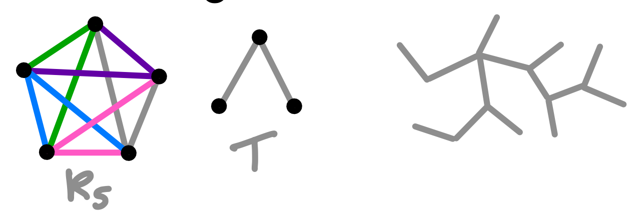

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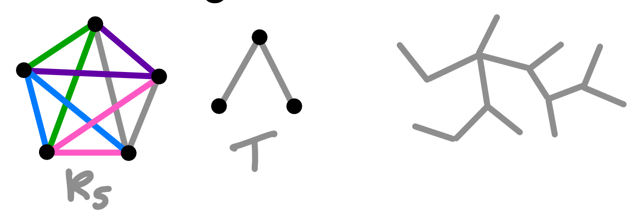
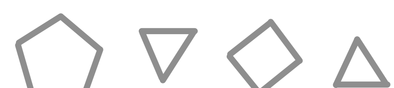


* <u>Oberwolfach problem (Ringel '67)</u>. Glock-Joos-Kim-Kühn-Osthus '21, Keevash-Staden '22. K_n decomposes into copies of F, for every 2-regular n-vx graph F and large odd n.



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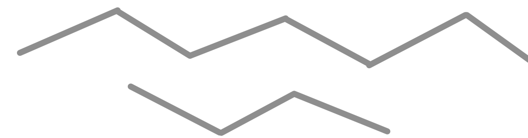

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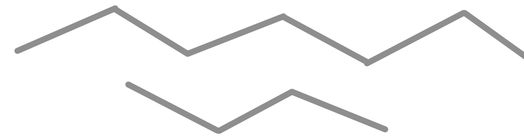

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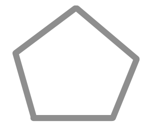

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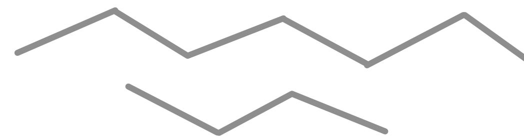

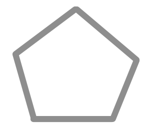

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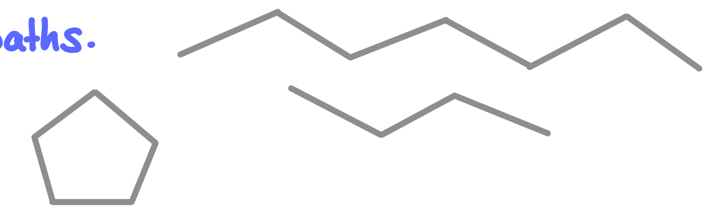

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  Allen-Böttcher-Clemens-Hladký-Piguet-Taraz '22+: true if $\Delta(T_i) \leq \frac{cn}{\log n}$.

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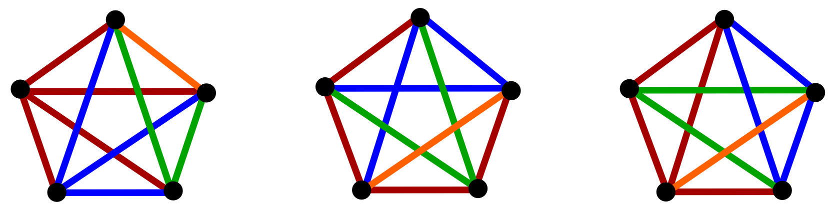

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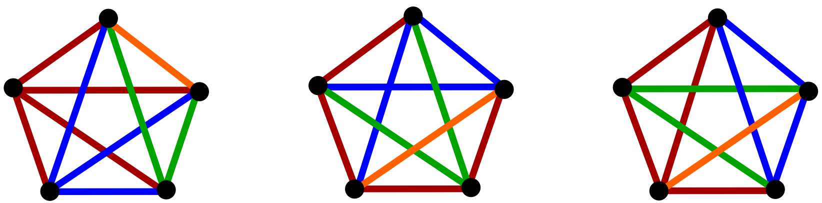

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* Some results for regular, complete multipartite, almost complete graphs.

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Theorem (Katsamaktsis–L.–Pokrovskiy–Sudakov 22+). Every star-forest with $\binom{m+1}{2}$ edges, whose i^{th} component has size $\geq \min\{1600i, 20m\}$, has an ASD into stars.

Proof plan

We will prove an approximate result:

Suppose: $e(G) = (1+\varepsilon)\binom{m+1}{2}$ and $\Delta(G) \leqslant cm$. Then G has a subgraph with $\binom{m+1}{2}$ edges which has an ASD.

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III) Obtain an ASD.

Define: $S = \{\text{vertices with } \deg < \frac{m}{10}\}$, $L = V(G) - S$.

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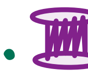
* Almost decompose edges in each set into $K_{t,t}$'s (t large constant). 

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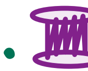
* Almost decompose edges in each set into $K_{t,t}$'s (t large constant). 

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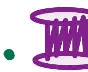
* Almost decompose edges in each set into $K_{t,t}$'s (t large constant). 

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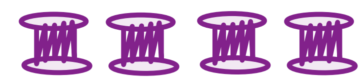
* Use Pippenger–Spencer '89 (about chromatic index of hypergraphs) to almost decompose $G[L]$ into $\leqslant \frac{2cm}{t}$ $K_{t,t}$-forests. 

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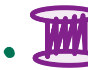
* Almost decompose edges in each set into $K_{t,t}$'s (t large constant). 

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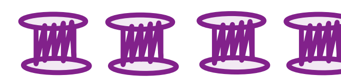
* Use Pippenger–Spencer '89 (about chromatic index of hypergraphs) to almost decompose $G[L]$ into $\leq \frac{2cm}{t}$ $K_{t,t}$-forests. 

* Rearrange to $\frac{m}{2}$ $K_{t,t}$-forests of equal size + small remainder.

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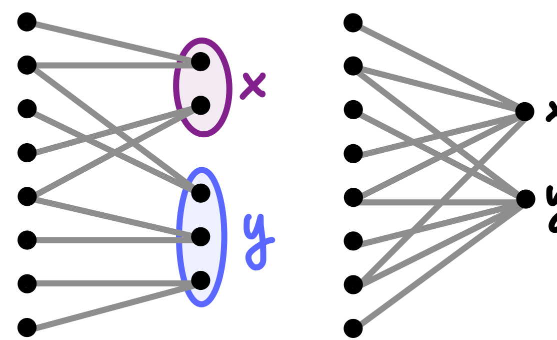

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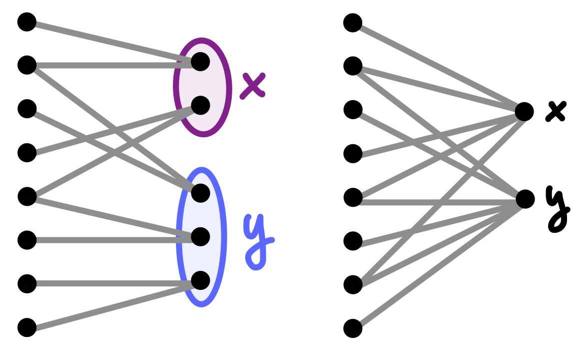

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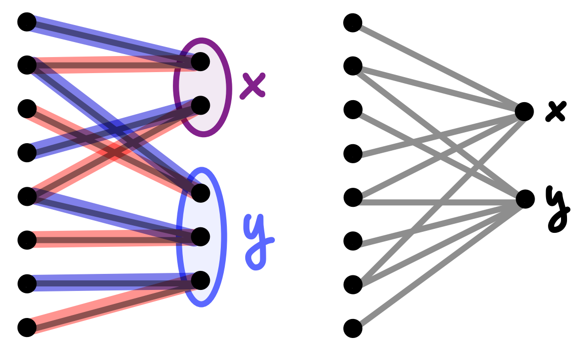

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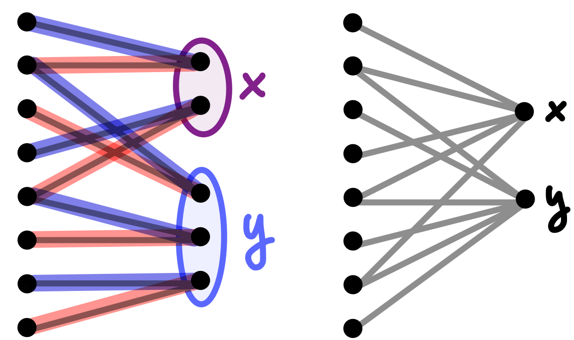

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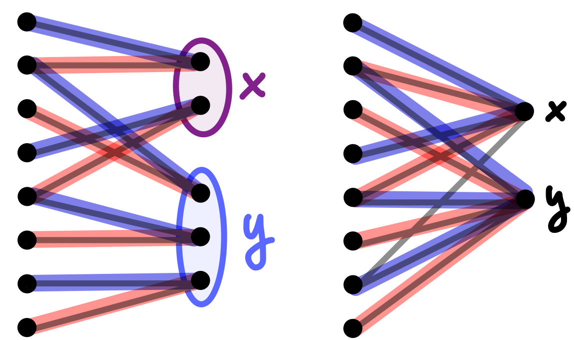

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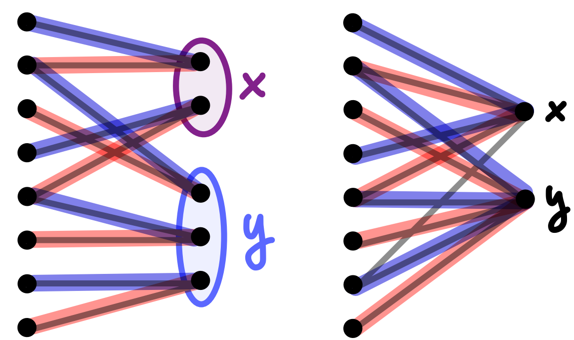

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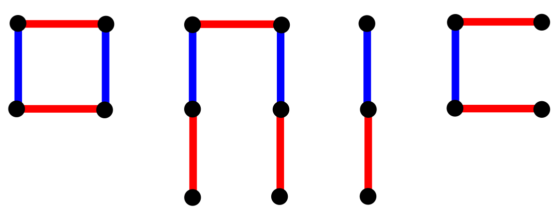

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We almost decomposed G into:

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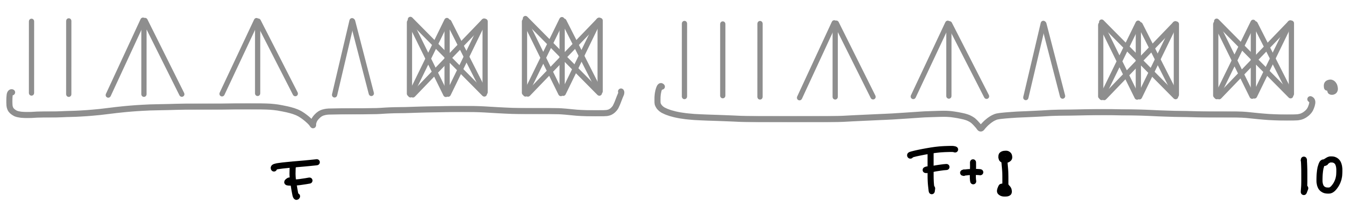
* Each $SF_{i,j} \cup KF_{5i+j}$ contains a copy of H, where $e(H) = m+1$ and

$H =$ 

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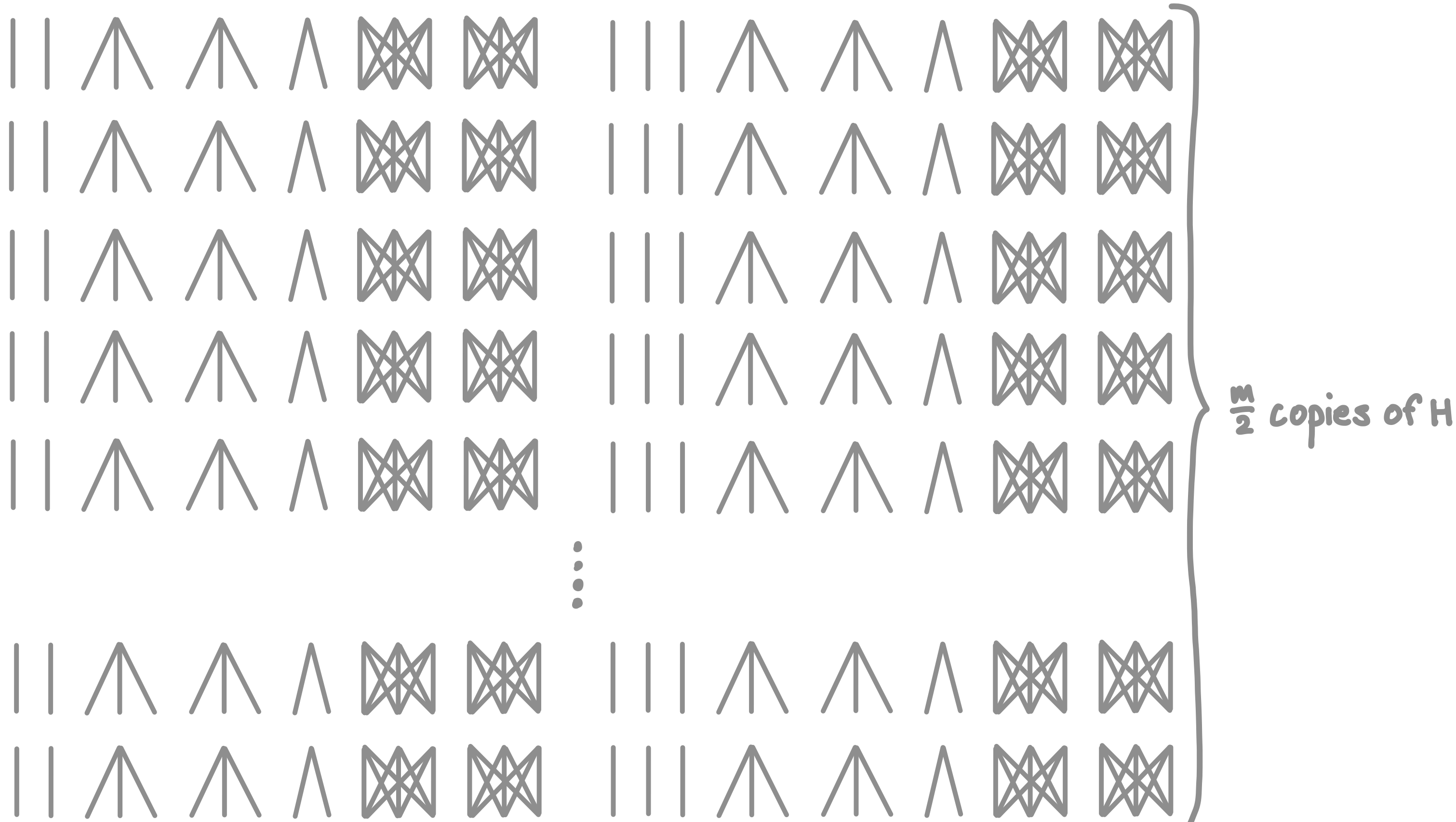
Step III: getting an ASD



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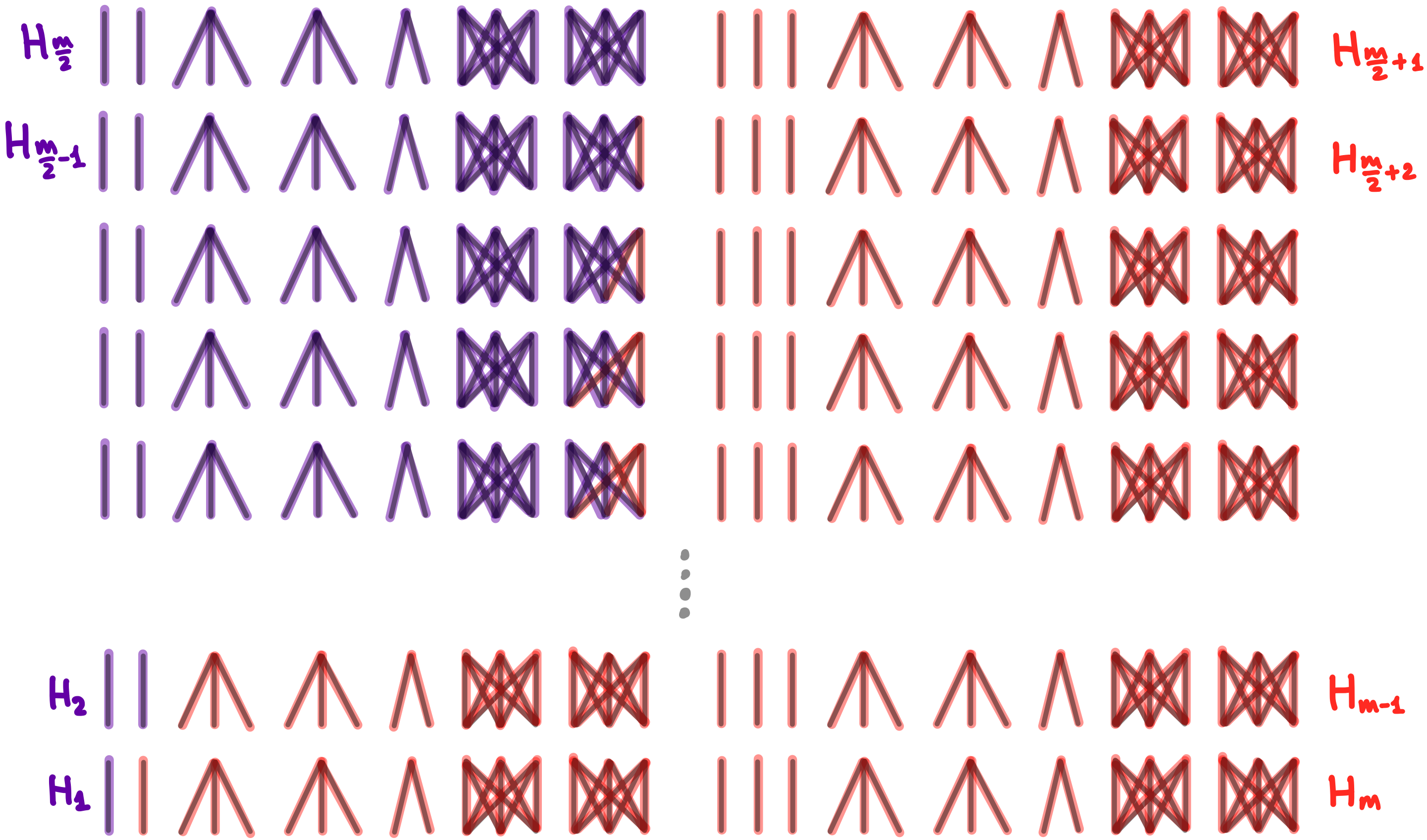

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Thank you for listening!