An improvement on Łuczak's connected matchings method

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Combinatorics Seminar

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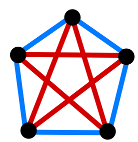
E.g. $K_5 \not\to K_3$ 

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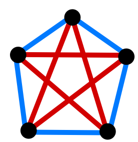
E.g. $K_5 \not\to K_3$ , but $K_6 \to K_3$.

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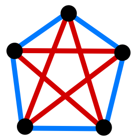
E.g. $K_5 \not\to K_3$ , but $K_6 \to K_3$.

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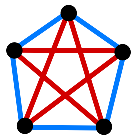
E.g. $K_5 \not\rightarrow K_3$ , but $K_6 \rightarrow K_3$. So $r(K_3) = 6$.

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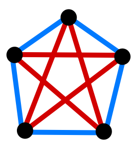
E.g. $K_5 \not\to K_3$ , but $K_6 \to K_3$. So $r(K_3) = 6$.

$G \xrightarrow{s} H$: in every s-colouring of G there is a mono copy of H.
The **s-colour Ramsey number** of H is $r_s(H) = \min\{N : K_N \xrightarrow{s} H\}$.

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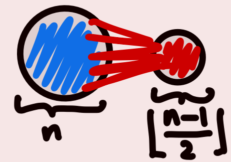
P_{n+1}: path of length n



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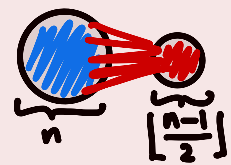

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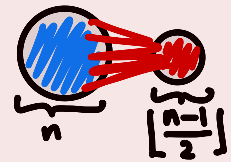

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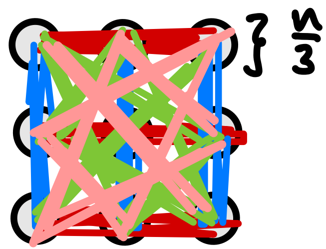
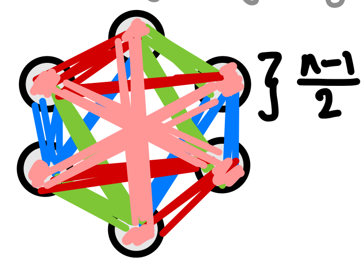


affine plane

Yongqi–Yuansheng–Feng–Bingxi '06



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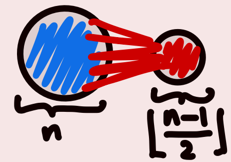

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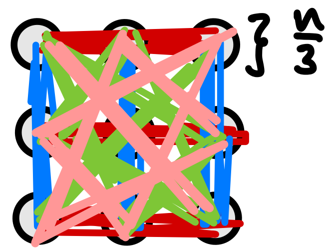

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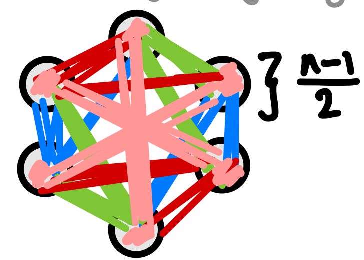

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Easy bounds on Ramsey numbers of paths

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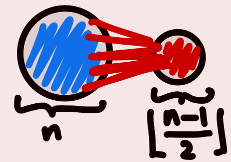

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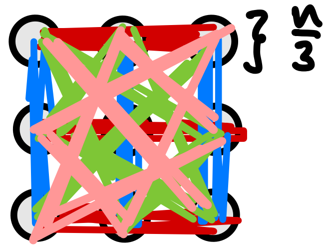

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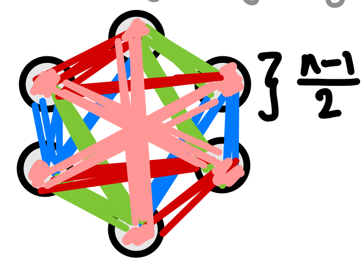

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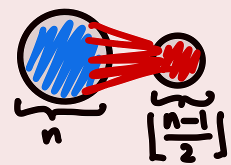

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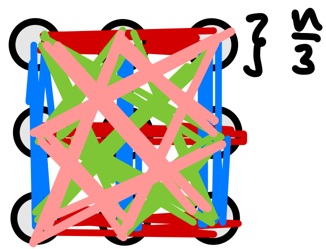

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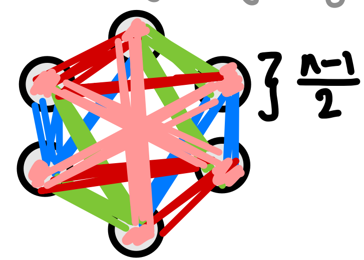

* $r_s(P_{n+1}) \lesssim sn+1$:

In an s-colouring of K_{sn+1}, the majority colour has average deg $\geq n \Rightarrow$ (Erdős–Gallai '59) it contains a P_{n+1}. □

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Gyarfás–Ruszinkó–Sárközy–Szemerédi '07: $r_3(P_{n+1}) = \begin{cases} 2n+1 & n \text{ even} \\ 2n & n \text{ odd}. \end{cases}$ (for large n)

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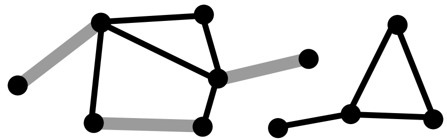

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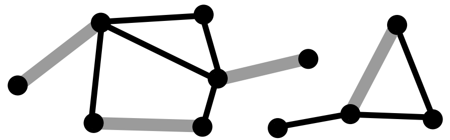

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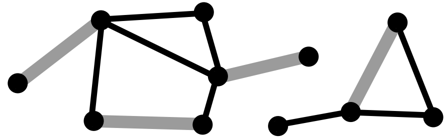

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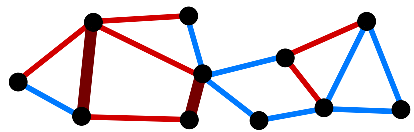
A mono connected matching is a matching contained in a mono connected component.



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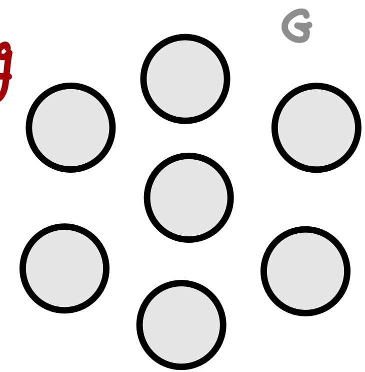
If $\forall \varepsilon > 0$, large n: $\forall$ "almost complete" G on $(\alpha + \varepsilon)n$ vs: $G \xrightarrow{s} CM(n)$,

Then $\forall$ large n: $r_s(P_n) \leq (\alpha + o(1))n$.

By <u>Szemerédi's regularity lemma</u>, given an s-colouring G of K_N, $\exists$ equipartition $\{V_1, \dots, V_k\}$ of the vs, where k is not-too-large-or-too-small, s.t. for almost all i,j the edges in each colour between V_i and V_j are "random-like".



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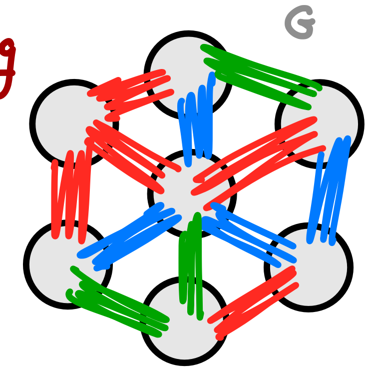

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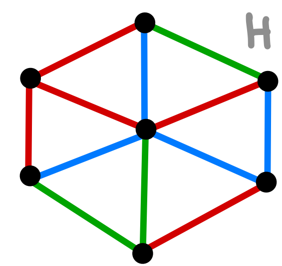

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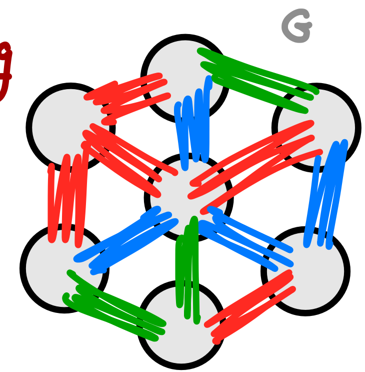

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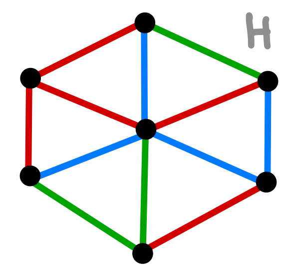

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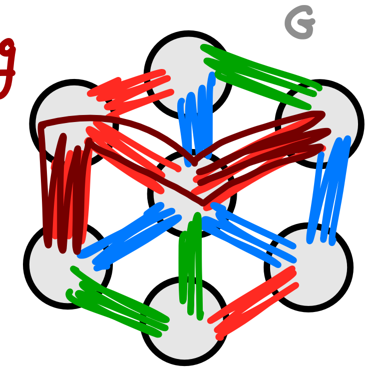

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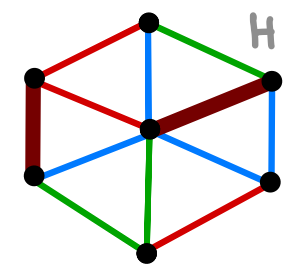

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* connected matchings in H on αk vs $\longleftrightarrow$ paths/cycles in G on $\approx \alpha n$ vs.

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If $\forall \varepsilon > 0$, large n: $\forall$ "almost complete" G on $(\alpha+\varepsilon)n$ vs: $G \xrightarrow{s} CM(n)$,

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Annoying! And cannot use induction ☹

Thm (Ł. 21'+). $\forall \varepsilon > 0$, large n: $K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n) \implies \forall n$: $r_s(P_n) \le (\alpha + o(1))n$.

Thm (L. 21'+). $\forall \varepsilon>0$, large n: $K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n) \Rightarrow \forall n: r_s(P_n) \leq (\alpha+o(1))n$.

We also prove similar results for

* asymmetric Ramsey numbers, (where different path length are required for different colours)
* cycles, (odd cycles require an additional condition.)
* complete bipartite / complete multipartite / blow-ups of small graphs,
* fractional connected matchings. (they admit more convenient structure theorems so might be better for applications)

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Our approach can be used to prove stability results. (e.g. showing that every r-colouring of K_N either contains a mono P_n or has special structure.)

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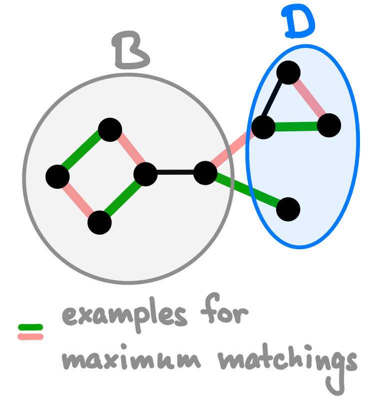
Our approach can be used to prove stability results. (e.g. showing that every r-colouring of K_N either contains a mono P_n or has special structure.)

With Bucić–Sudakov '19 we proved a version for $K_{n,n}$.

Given graph G, let $\underline{B} = \{$vs in every maximum matching$\}$

$\underline{D} = V(G) \setminus B$



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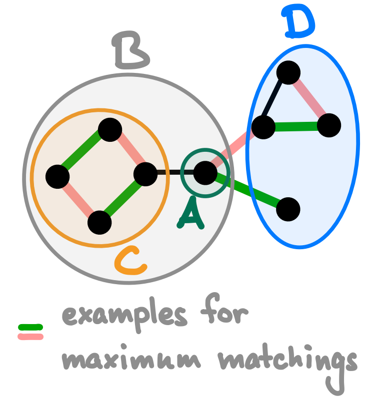

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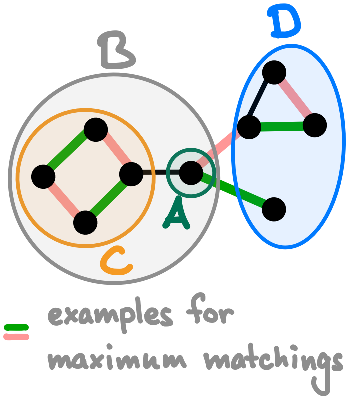

Gallai-Edmonds: For every maximum matching M:

* $M[C]$ is a perfect matching in $G[C]$,
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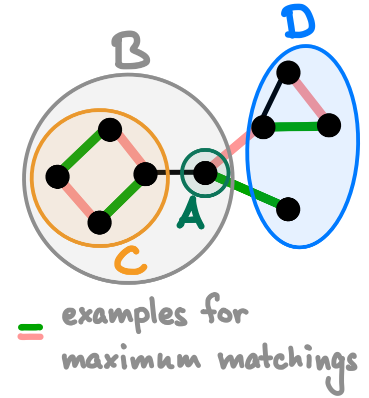

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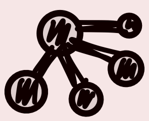
Lem. Let G be maximal on n vs with no matching of size m. Then G is a complete blow-up of a star.



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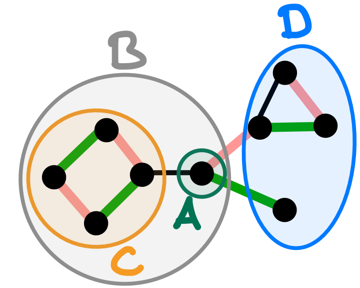

examples for maximum matchings

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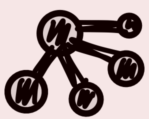
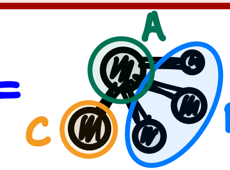


Proof sketch. Let A, C, D be as above. By maximality, $G =$  $\square$

Proving the theorem

Thm ($\mathcal{L}$. 21'+). $\forall \varepsilon > 0$, large n: $K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n) \implies \forall n$: $r_s(P_n) \leq (\alpha + o(1))n$.

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Lem (Figaj–Łuczak '07).
If $\forall \varepsilon > 0$, large n: $\forall$ "almost complete" G on $(\alpha+\varepsilon)n$ vs: $G \xrightarrow{s} CM(n)$,
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To prove the theorem, it suffices to prove the following:

If $K_N \xrightarrow{s} CM(n)$,

then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $|G| = N + \varepsilon n$ and every vx in G has $\le \delta n$ non-neighbours then $G \xrightarrow{s} CM(n)$.

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(could add c-coloured edges between two small compts)

* Every mono compt $\mathcal{U}$ is a complete blow-up of a star.
(by key lemma)

Start of proof

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* $|G| = N+\varepsilon n$, every vx in G has $\le \delta n$ non-neighs.
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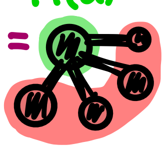
* Every mono compt $\mathcal{U}$ is a complete blow-up of a star. Write: $\mathcal{U} =$ 

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End of proof (modulo claim)

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$\Rightarrow$ Contradiction to $K_N \xrightarrow{s} CM(n)$! □

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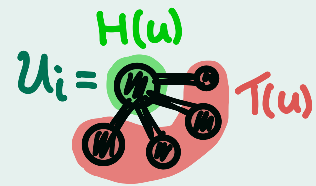

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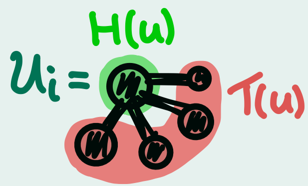

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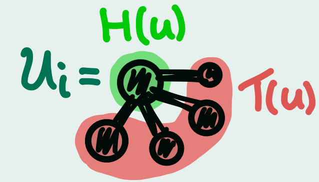

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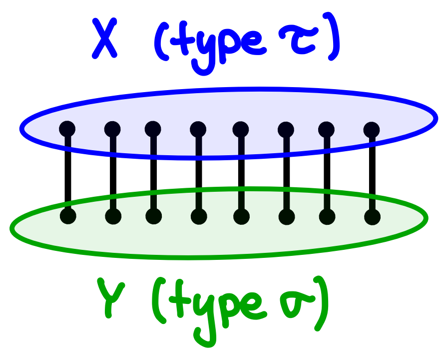
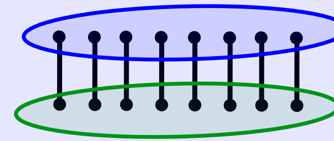
$\Rightarrow \exists$ types $\underline{\tau, \sigma}$ and $\underline{M_0} \subseteq M$:

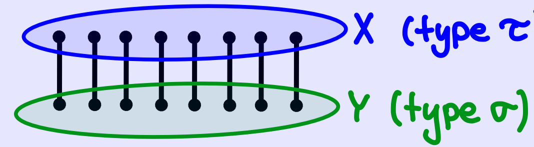
* edges in M_0 have ends of types τ, σ,
* $|M_0| \geq 4^s \delta n$. (as #types is small and $\delta \ll \varepsilon$)



Plan: find $M_0 \supseteq M_1 \supseteq \dots \supseteq M_s$:

* $|M_i| \geq \frac{1}{4} \cdot |M_{i-1}|$
* $\nexists$ no i-coloured edges between $X_i = X \cap V(M_i)$ and $Y_i = Y \cap V(M_i)$.





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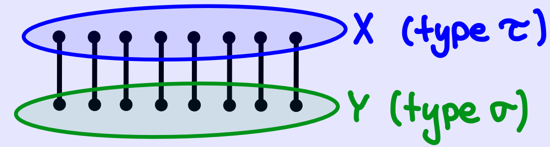

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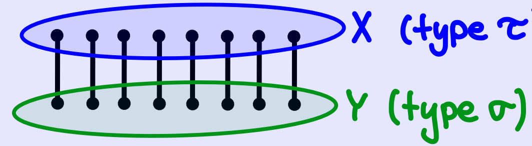

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Write $\tau = (u_1, \dots, u_s, \alpha_1, \dots, \alpha_s)$ and $\sigma = (w_1, \dots, w_s, \beta_1, \dots, \beta_s)$.

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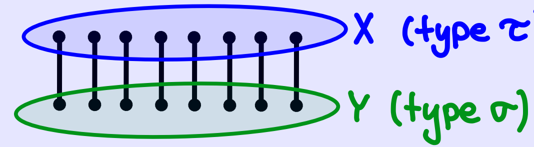

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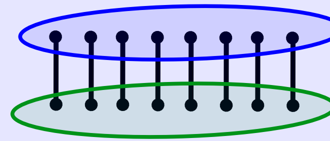

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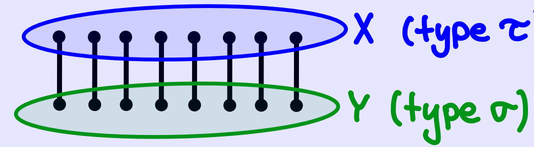
(1) $u_i \neq w_i$

Plan: find $M_0 \supseteq M_1 \supseteq \dots \supseteq M_s$:



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Write $\tau = (u_1, \dots, u_s, \alpha_1, \dots, \alpha_s)$ and $\sigma = (w_1, \dots, w_s, \beta_1, \dots, \beta_s)$.

Suppose $M_0 \supseteq M_1 \supseteq \dots \supseteq M_{i-1}$ are as above.

1. $u_i \neq w_i \Rightarrow X_{i-1}$ and Y_{i-1} are in distinct i-coloured compts.

Plan: find $M_0 \supseteq M_1 \supseteq \dots \supseteq M_s$:

* $|M_i| \geqslant \frac{1}{4} \cdot |M_{i-1}|$
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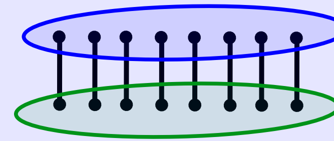


X (type τ)

Y (type σ)

$\Rightarrow |X_s|, |Y_s| \geqslant 4^{-s} \cdot \overbrace{|M_0|}^{>4^s \cdot \delta n} > \delta n$, $G_1[X_s, Y_s]$ is empty.

contradiction to: every vx in G_1 has $\leq \delta n$ non-neighs. $\square$

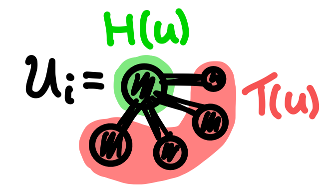
Write $\tau = (U_1, \dots, U_s, \alpha_1, \dots, \alpha_s)$ and $\sigma = (W_1, \dots, W_s, \beta_1, \dots, \beta_s)$.

Suppose $M_0 \supseteq M_1 \supseteq \dots \supseteq M_{i-1}$ are as above.

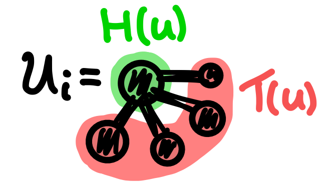
① $U_i \neq W_i \Rightarrow X_{i-1}$ and Y_{i-1} are in distinct i-coloured compts.

$\Rightarrow$ can take $M_i = M_{i-1}$.

(2) $\mathcal{U}_i = W_i$



(2) $\mathcal{U}_i = W_i \Rightarrow \alpha_i = \beta_i = T$, (otherwise, if say $\alpha_i = H$ then X_{i-1} is joined to all of $\mathcal{U}_i \geq Y_{i-1}$, contradiction to $M_{i-1} \subseteq \overline{G_1}$.)



② $\mathcal{U}_i = W_i \Rightarrow \alpha_i = \beta_i = T$,
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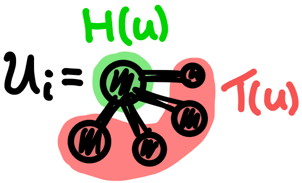

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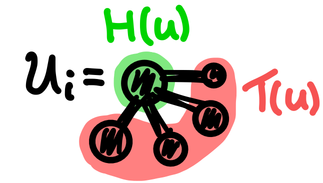
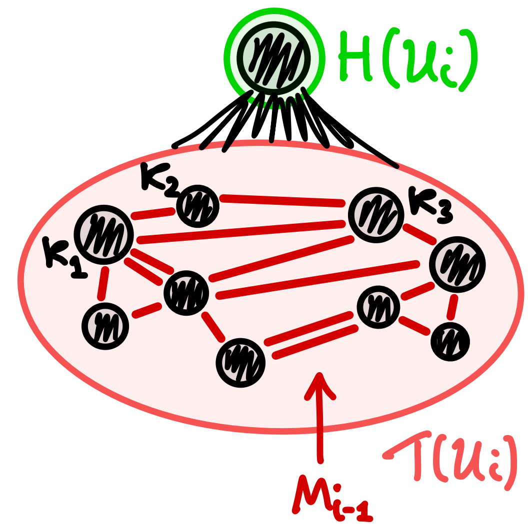


Let $\underline{K_1, \dots, K_\ell}$ be the cliques in $T(\mathcal{U}_i)$.



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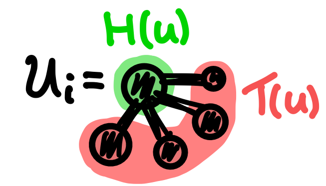
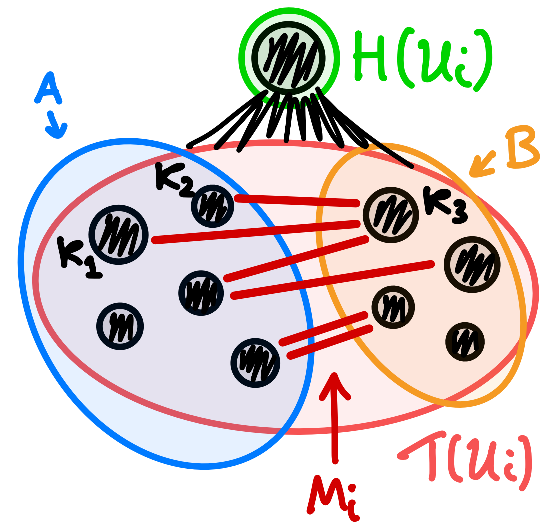


Let $\underline{K_1, \dots, K_\ell}$ be the cliques in $T(\mathcal{U}_i)$.

Take $\{A, B\}$ to be a random partition of $[\ell]$.

$$M_i = \left\{ xy \in M_{i-1} : x \in X_{i-1} \cap \left(\bigcup_{j \in A} K_j \right),\ y \in Y_{i-1} \cap \left(\bigcup_{j \in B} K_j \right) \right\}.$$



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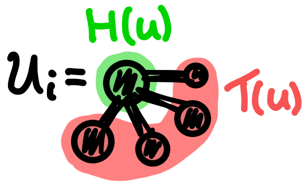

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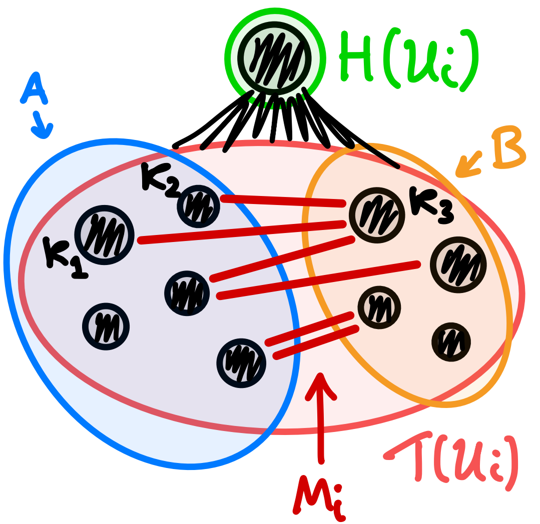
$\forall e \in M_{i-1} : \mathbb{P}(e \in M_i) = 1/4$.

(if $xy \in M_{i-1}$ then $x \in K_s$, $y \in K_t$ for some $s \neq t$.
$\mathbb{P}(xy \in M_i) = \mathbb{P}(s \in A, t \in B) = \frac{1}{4}$)



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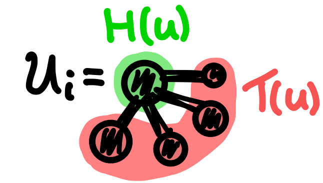

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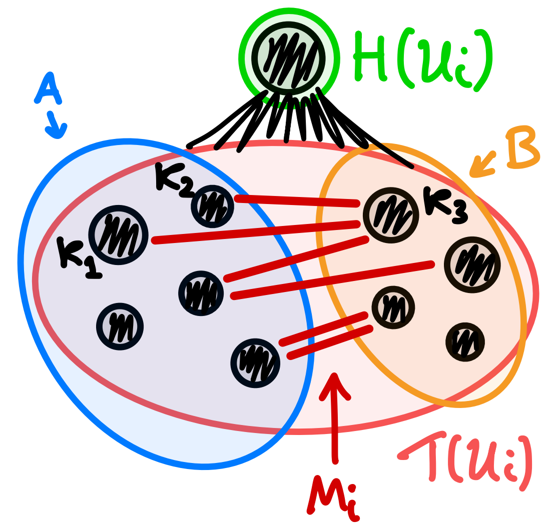

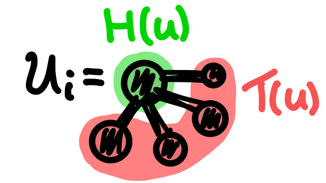

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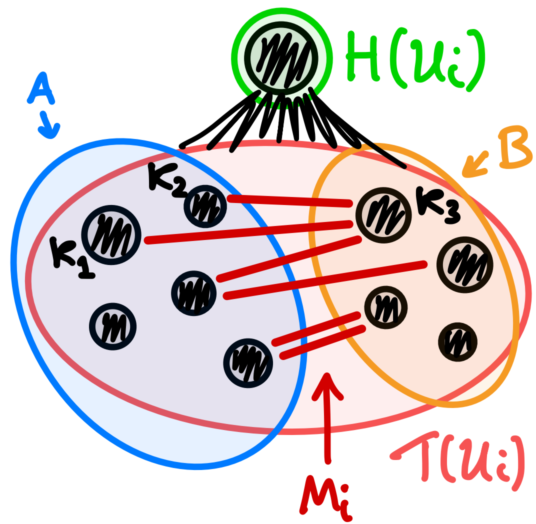
$\Rightarrow \mathbb{E}[|M_i|] = \frac{1}{4} \cdot |M_{i-1}| \Rightarrow$ appropriate M_i exists. $\square$



Thm (Ł. 21'+). $\forall \varepsilon>0$, large n: $K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n) \implies \forall n$: $r_s(P_n) \le (\alpha+o(1))n$.

Thm (L. 21'+). $\forall \varepsilon > 0$, large n: $K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n) \implies \forall n$: $r_s(P_n) \leq (\alpha + o(1))n$.

* as mentioned, we prove a more general result.

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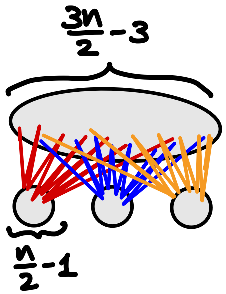
  - Bucić–Ł.–Sudakov '19. By a version of above + induction, $K_{N,N} \xrightarrow{3} P_n$ for $N \approx \frac{3n}{2}$.



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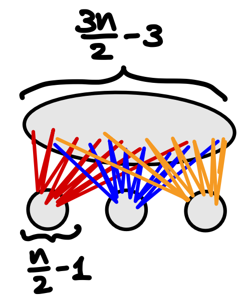

- By above + induction, easier proof for $r_3(P_n) \approx 2n$.

Summary

Thm ($\mathcal{L}$. 21'+). $\forall \varepsilon > 0$, large n: $K_{(\alpha+\varepsilon)n} \xrightarrow{s} CM(n) \Rightarrow \forall n$: $r_s(P_n) \leq (\alpha + o(1))n$.

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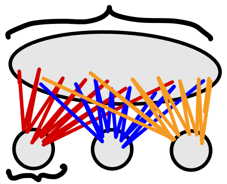

* Open problem. Determine $r_s(P_n)$. (best bounds for $s \geq 4$: $(s-1)n \lesssim r_s(P_n) \lesssim (s - \frac{1}{2})n$)

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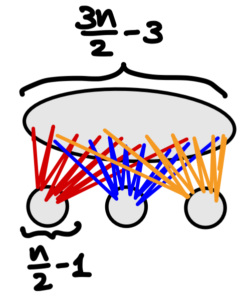

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Thank you for listening!