Digraph immersions

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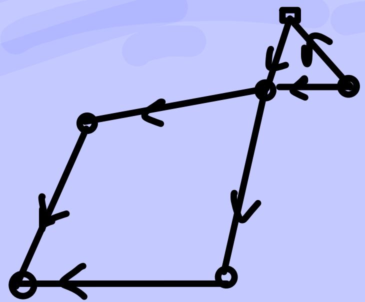
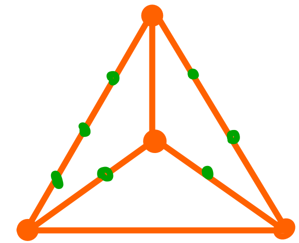


Joint work with António Girão

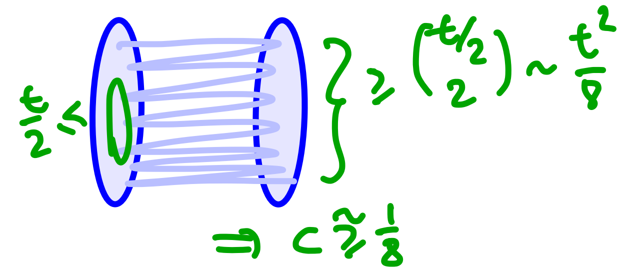
Clique subdivisions

Bollobás–Thomason / Komlós–Szemerédi '96:

$\exists c>0$ s.t. if G has $\overset{\text{min}}{\text{average}}$ deg $\geq ct^2$ then G has a subdivision of K_t.



Tight (up to a constant factor): $G = K_{k,k}$

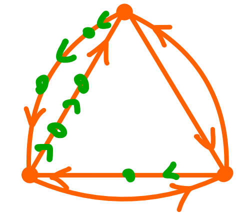


Best known bounds on least c: $\geq \frac{9}{64}$ (Łuczak)

$\leq \frac{10}{23}$ (Kühn–Osthus '06)

What about digraphs?

$\vec{K}_t$ complete digraph on t vertices.

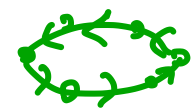


$\vec{K}_3$

① Is there $f(t)$ s.t.: if G has min in & out-deg $\geq f(t)$ then G contains a subdivision of $\vec{K}_t$?

- Yes for $t=2$.

  min out-deg $\geq 1 \Rightarrow$ subdivision of $\vec{K}_2$.

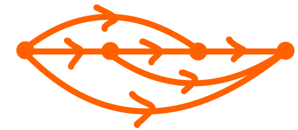
  $f(2)=1$

- No for $t \geq 3$

  (Mader '85 using construction of Thomassen 85'

  DeVos-McDonald-Mohar-Scheide '12).

TT_t transitive tournament on t vertices.



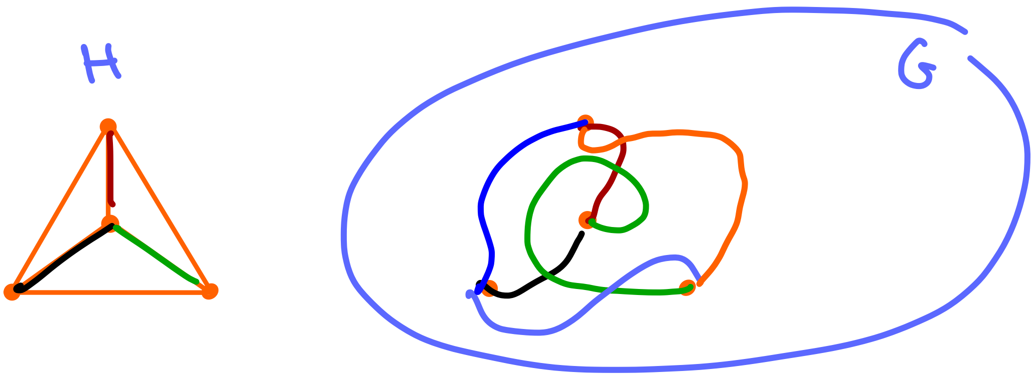
TT_4

② Is there $f(t)$ s.t.: if G has min in & out deg $\geq f(t)$ then G contains a subdivision of TT_t?

This is (almost) a conjecture of Mader '96.

Immersions

G <u>immerses</u> H if $\exists$ injection $f: V(H) \to V(G)$ and edge-disjoint paths P_{uv}, for $uv \in E(H)$, s.t. P_{uv} starts at $f(u)$ and ends at $f(v)$.



DeVos–Dvořák–Fox–McDonald–Mohar–Scheide '14:

If G has average deg $\geq \underset{t-1}{200t}$ then G immerses K_t.

K_{t-1} does not immerse K_t

Dvořák–Yepremyan '17: min deg $\geq 11t+7 \Rightarrow$ immersion of K_t.

Hong–Wang–Yang '20: average deg $\geq (1+\varepsilon)t$ & H-free for H bipartite $\Rightarrow$ immersion of K_t.

Conjecture (Lescure–Meyniel '89): G immerses $K_{\chi(G)}$.

Immersion in digraphs

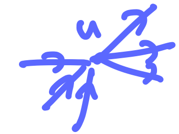
③ Is there $f(t)$ s.t.: if G has min in & out-deg $\geq f(t)$ then G immerses $\vec{K}_t$?

No for $t \geq 3$ (DMMS '12).

Lochet '19: $\exists f(t)$ s.t.: if G has min out-deg $\geq \underbrace{f(t)}_{ct^3}$ then it immerses TT_t.

Conceivable that can take $f(t) = ct$.

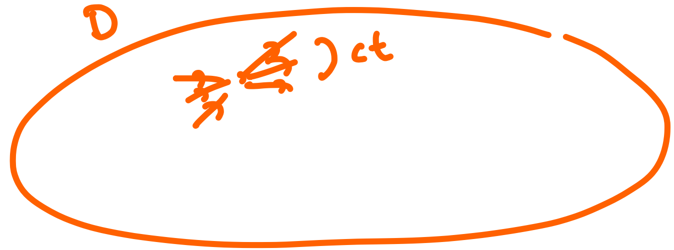
G is **Eulerian** if $d^+(u) = d^-(u)$ for every vertex u.



DeVos–McDonald–Mohar–Scheide '12: If G is Eulerian with min out-deg $\geq t^2$ then it immerses $\vec{K}_t$.

Thm (Girão–L. 22+). $\exists c > 0$ s.t. if G is Eulerian with min out-deg at least ct then G immerses $\vec{K}_t$.

Overview of the proof.



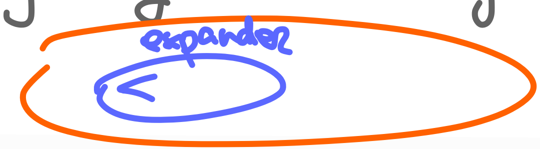
I) Let c be a large constant.

Lemma. D Eulerian with min out-deg $\geq ct$ $\Rightarrow$ D immerses a digraph G with $\Theta(t)$ vertices and $\Omega(t^2)$ edges.

We use 'sparse expanders'.

* Introduced by Komlós–Szemerédi '96.

* Can be found in graphs with average deg at least a large constant. G $d(G) \geq 100$



* many recent applications:

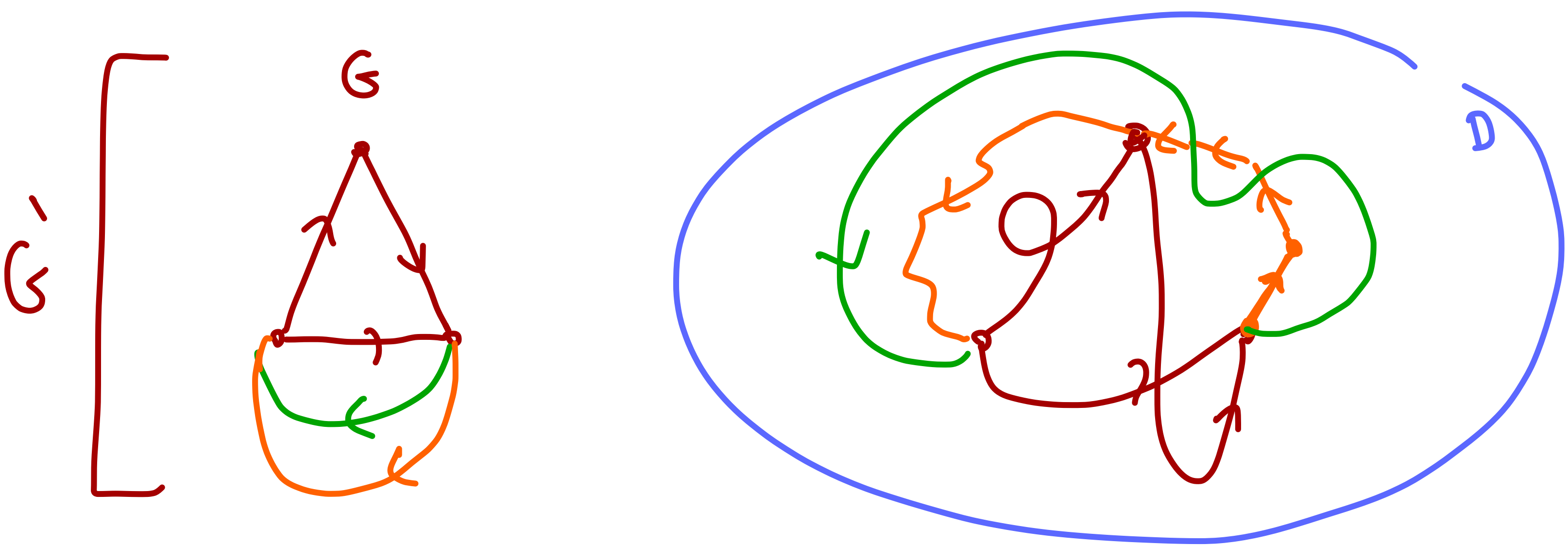
  - odd cycle problem (Liu–Montgomery '20+)

  - clique subdivisions in C_4-free graphs (Liu–Montgomery '17)

  - Komlós conjecture on Hamiltonian sets (Kim–Liu–Sharifzadeh–Staden '17)

Our proof is a rare use of expanders in digraphs (Eulerian + immersion help).

II) **Observation:** If D is Eulerian and immerses G then it immerses an Eulerian multidigraph $G' \supseteq G$ with $V(G') = V(G)$.



III) **Lemma.** If G' is an Eulerian multidigraph on n vertices whose underlying graph (obtained by removing directions and multiplicities) has min deg $\geq \alpha n$, then it immerses $\overrightarrow{K}_s$, where $s = c'\alpha^{-4} n$.

I) + II) + III) $\Longrightarrow$ theorem.

D Eulerian, $\delta^+ \geq ct$.

I) $D \rightsquigarrow G$, G has $\Theta(t)$ vertices, $\geq c' t^2$ edges.

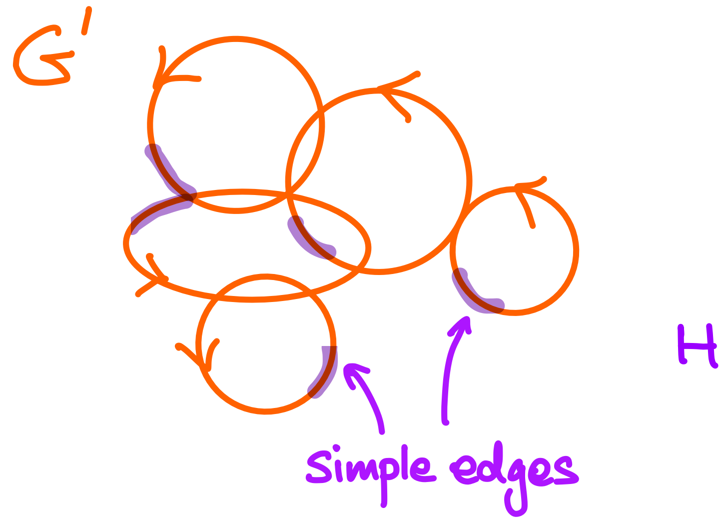
$\downarrow$ immersion

II) $D \rightsquigarrow G'$, G' Eulerian, multidigraph $G' \supseteq G$, $V(G') = V(G)$

III) $G' \rightsquigarrow \overrightarrow{K}_t$. $\Longrightarrow$ $D \rightsquigarrow \overrightarrow{K}_t$.

More about III

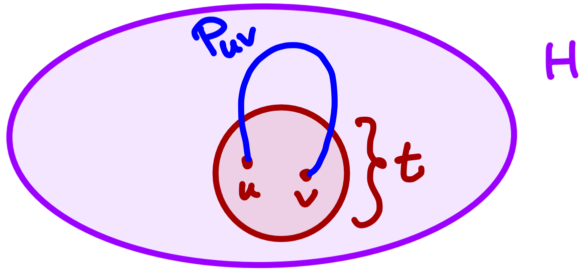
We will find a collection $\mathcal{C}$ of $\Omega(n^2)$ edge-disjoint dicycles in G' each containing an edge which is simple in $\bigcup_{C \in \mathcal{C}} C$.



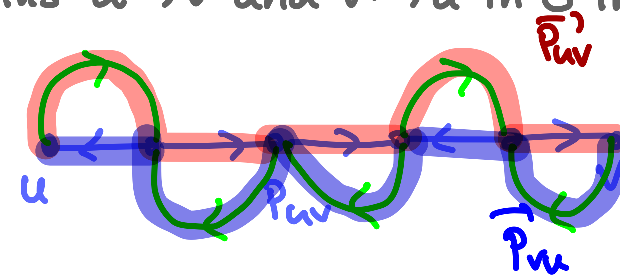
Let H be the undirected graph formed by the purple (simple) edges.

Then H has average deg $\Omega(n)$.

Thus, by **DDFMMS '14**: H immerses K_t, where $t = \Omega(n)$.



Each P_{uv} corresponds to paths $u \to v$ and $v \to u$ in G' that are edge-disjoint.



$\Rightarrow$ G' immerses $\overrightarrow{K}_t$.

Open problems

① What is min $f(t)$ s.t. if $\delta^+(G) \geq f(t)$ then G immerses TT_t?

Lochet '19: $f(t) = O(t^3)$.

Maybe $f(t) = O(t)$?

② **Mader '96**: Is there $g(t)$ s.t. if $\delta^+(G) \geq g(t)$ then G contains a subdivision of TT_t?

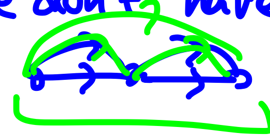
③ **Conjecture (Lescure–Meyniel '89)**: G immerses $K_{\chi(G)}$.

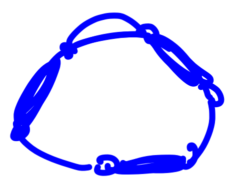
Finding $\mathcal{C}$

Lemma 1. D multigraph on n vertices with min out-deg $\geq \alpha n$.
Then $\exists$ dicycle with $\leq \frac{4}{\alpha}$ simple edges.

Apply Lemma 1 repeatedly to find $\mathcal{C}$.

preprocessing of G' that ensures that we don't have a dipath of length 2 with two multiple edges.





immersion.

G immerses H iff H can be obtained from G by:

* delete edge / vx

* or replace a path uvw by uw.

Proof of Lemma 1

$\delta^+(u) \geq 2n$

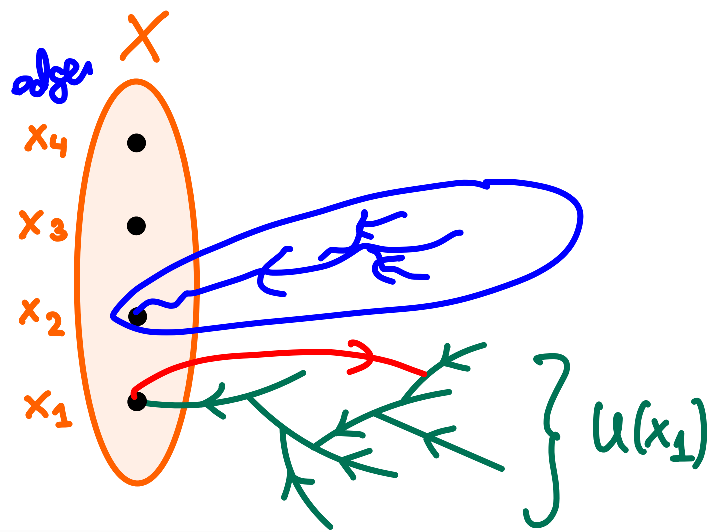
Lemma 2. D digraph, $\omega : V(D) \to \mathbb{R}^+$. If $\omega(N^+(u)) \geq \alpha \cdot \omega(V(D))$ then $\exists$ dicycle of length $\leq \frac{4}{\alpha}$.

$D' \subseteq D$ simple subdigraph, $E(D') = \{ xy : xy \text{ is a multiple edge in } D \}$.

$X = \{ x \in V(D') : d^+_{D'}(x) = 0 \}$.

$X = \emptyset$
$\Rightarrow \exists$ cycle
w/ no simple edge

Find sets $U(x)$, $x \in X$, as follows:



If $\exists$ edge $x \xrightarrow{D} U(x)$, then $\exists$ dicycle in D with ≤ 1 simple edge.

Suppose $\exists$ no such edges.

D_0 digraph on X with $x \to y$ iff $\exists$ edge $x \xrightarrow{D} U(y)$.

Can check: $\omega\left(N^+_{D_0}(x)\right) \geq \alpha\, \omega(X)$ $\forall x \in X$, where $w(x) = |U(x)|$.

By Lemma 2: $\exists$ dicycle of length $\leq \frac{4}{\alpha}$ in D_0.

$\Rightarrow \exists$ dicycle in D with $\leq \frac{4}{\alpha}$ simple edges.