Hypergraphs with no tight cycles

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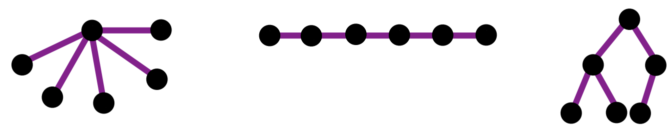
July 2021

Introduction

Fact. The max number of edges in a graph on n vertices with no cycles is $n-1$.

(Maximisers are trees.)



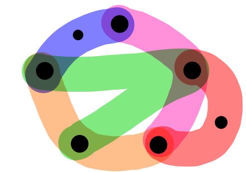
An **r-graph** (= r-uniform hypergraph) is a hypergraph whose edges consist of r vertices.

Question. What is the maximum possible number of edges in an r-graph on n vertices which has no cycles?

The answer depends on the definition of a cycle.

We consider three notions: Berge cycles, loose cycles and tight cycles.

Berge cycles

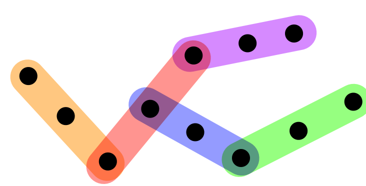


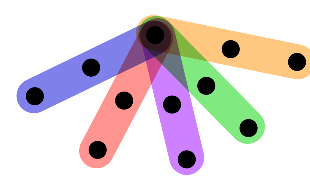
A **Berge cycle** is a sequence $e_1, v_1, \ldots, e_k, v_k$ where $e_1, \ldots, e_k$ are distinct edges, $v_1, \ldots, v_k$ are distinct vertices, and $v_i \in e_i \cap e_{i+1}$ for $i = 1, \ldots, k-1$ and $v_k \in e_k \cap e_1$.

Claim. The maximum number of edges in an r-graph on n vertices with no Berge cycles is $\lfloor \frac{n-1}{r-1} \rfloor$.

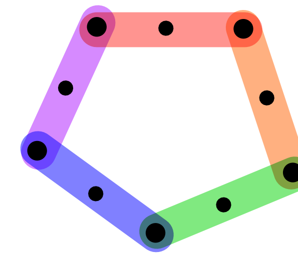
Proof of upper bound. Not hard to show: no Berge cycle $\Rightarrow$ edges can be ordered as $e_1, \ldots, e_m$ s.t. $|e_i \cap (e_1 \cup \ldots \cup e_{i-1})| \leq 1$ $\forall i$.

Proof of lower bound.





Loose cycles



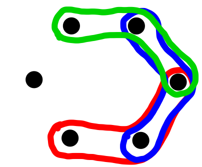
A **loose cycle** is a sequence $e_1, \ldots, e_k$ of edges s.t. consecutive pairs of edges (and e_k and e_1) have exactly one vertex in common, and non-consecutive pairs are disjoint.

Theorem (Frankl–Füredi '87). The maximum number of edges in an r-graph on n vertices with no loose cycles is $\binom{n-1}{r-1}$ (for n large).

Actually, FF show that $> \binom{n-1}{r-1}$ edges $\Rightarrow \exists$ loose triangle.
(cycle of length 3)

Proof of lower bound. Consider the r-graph with vertices $[n] := \{1, \ldots, n\}$ whose edges are all sets of r vertices containing 1.

Tight cycles.



A **tight cycle** is a sequence $v_1, \ldots, v_k$ of distinct vertices s.t. $\{v_i, \ldots, v_{i+r-1}\}$ is an edge $\forall i$ (indices are modulo k).

$f_r(n)$ = max number of edges in an r-graph on n vertices with no tight cycles.

Lower bounds

Observation. $f_r(n) \geq \binom{n-1}{r-1}$.

Proof. Consider r-graph H with $V(H)=[n]$ and $E(H)=\{r\text{-sets containing } 1\}$.

Conjecture (Sós, Verstraëte '10s). $f_r(n)=\binom{n-1}{r-1}$ for large n.

Huang–Ma '19. The conjecture is false (for $r\geq 3$).
There exists $c\in(1,2)$ s.t. $f_r(n)\geq c\binom{n-1}{r-1}$.

B. Janzer '20. Let $r\geq 3$. There exists $c>0$ s.t. $f_r(n)\geq \dfrac{cn^{r-1}\log n}{\log\log n}$.

Upper bounds

A result of Erdős ('64) implies $f_r(n) = O(n^{r-2^{-(r-1)}})$.

An unpublished result of Verstraëte implies $f_3(n) = O(n^{3/2})$.

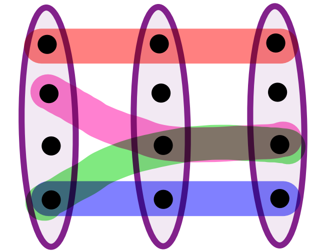
Sudakov-Tomon ('21). Let $r \geq 3$. Then $f_r(n) \leq n^{r-1} e^{O(\sqrt{\log n})}$.

In particular, $f_r(n) = n^{r-1+o(1)}$.

Theorem (L. '21+). Let $r \geq 3$. Then $f_r(n) \leq O(n^{r-1}(\log n)^5)$.

(This is tight up to a factor of $O((\log n)^4 \log\log n)$.)

r-partite r-graphs



Claim. Let $\mathcal{H}$ be an r-graph on m edges. There is an r-partite subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ on $\geq \frac{r!}{r^r} \cdot m$ edges.

Proof. Let $\{A_1, \ldots, A_r\}$ be a partition of $V(\mathcal{H})$, chosen uniformly at random, and let $\mathcal{H}'$ be the subhypergraph of $\mathcal{H}$ obtained by keeping edges e with $|e \cap A_i| = 1$ $\forall i$.

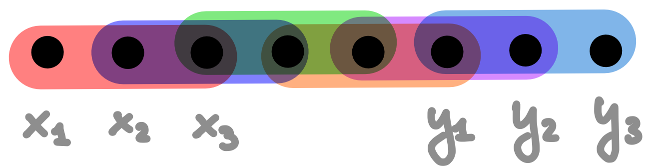
For $e \in \mathcal{H}$, $\mathbb{P}(e \in \mathcal{H}') = \frac{r!}{r^r}$ $\Rightarrow$ $\mathbb{E}(e(\mathcal{H}')) = \frac{r!}{r^r} \cdot m$ $\Rightarrow$ suitable $\mathcal{H}'$ exists.

Aim. Let $\mathcal{H}$ be an r-partite r-graph, with partition $\{A_1, \ldots, A_r\}$, on n vertices with $\geq d n^{r-1}$ edges, where $d \geq c \cdot (\log n)^5$. Then $\mathcal{H}$ contains a tight cycle.

Notation

We represent edges $e \in \mathcal{H}$ as $(x_1, \ldots, x_r)$ where $x_i \in A_i$.

For edges $e = (x_1, \ldots, x_r)$ and $f = (y_1, \ldots, y_r)$, a <u>tight path from e to f</u> is a tight path $(z_1, \ldots, z_\ell)$ where $z_i = x_i$ for $i \in [r]$ and $z_{\ell - r + i} = y_i$ for $i \in [r]$.



Expanders

r-partite r-graph, $\geq dn^{r-1}$ edges

Lemma (Sudakov–Tomon '21). There is a subhypergraph $\mathcal{G} \subseteq \mathcal{H}$ s.t. for every edge $e \in \mathcal{G}$ and small $B \subseteq V(\mathcal{G})$ (which is disjoint of e): for almost every edge $f \in \mathcal{G}$ there is a short tight path from e to f that avoids B.

Proof ideas.

* Define r-line graphs, which are graphs associated with r-partite r-graphs.
* Define notions of density and expansion for r-line-graphs.

Call such $\mathcal{G}$ an expander.

Connecting edges in expanders.

<u>Theorem (Sudakov–Tomon '21)</u>. Let $\mathcal{G}$ be an expander. Then

* Either for every disjoint edges e, f there is a tight path from e to f,
* Or there is a small dense subhypergraph.

They proceed by a density increment argument.

<u>Theorem (L. 21+)</u>. Let $\mathcal{G}$ be an expander. Then $\mathcal{G}$ contains a tight cycle.

In fact, we show that there is a tight cycle through almost every two edges.

Robust reachability

Let $\mathcal{G}$ be an expander with m edges. So, for every $e \in E(\mathcal{G})$ and $B \subseteq V(\mathcal{G})$ with $|B| \leq \frac{\ell^2}{\varepsilon}$, which is disjoint of e, for at least $(1-\varepsilon)m$ edges $f \in E(\mathcal{G})$: there is a tight path from e to f of length $\leq \ell$ that avoids B.

Lemma (L.). In $\mathcal{G}$ as above, for every edge e there is a set $F(e) \subseteq E(\mathcal{G})$ with $|F(e)| \geq (1-\varepsilon)m$, and paths $P(e,f)$, for $f \in E_e$, of length $\leq \ell$ from e to f, s.t. no vertex (except for those in e) appears in $> \frac{\varepsilon m}{\ell}$ paths $P(e,f)$.

Proof of lemma

* Let $F(e) \subseteq E(G)$ be maximal s.t. $\exists$ paths $P(e,f)$ of length $\le \ell$ from e to f, for $f \in E_e$, s.t. no vertex appears in $> \frac{\varepsilon m}{\ell}$ paths $P(e,f)$.

* Let B be the set of vertices in exactly $\frac{\varepsilon m}{\ell}$ paths $P(e,f)$.

Then $|B| \cdot \frac{\varepsilon m}{\ell} \le \ell \cdot |F(e)| \le \ell m \Rightarrow |B| \le \frac{\ell^2}{\varepsilon}$.

* By expansion, there is a subset $F'(e) \subseteq E(G)$ of size $\geq (1-\varepsilon)m$, and paths $P'(e,f)$ for $f \in F'(e)$, of length $\le \ell$ from e to f, that avoids B.

* If $|F(e)| < (1-\varepsilon)m$, let $f' \in F'(e) \setminus F(e)$.

The set $F(e) \cup \{f'\}$ with paths $P(e,f)$ for $f \in F(e)$ and $P'(e,f')$ for f', contradicts maximality of $F(e)$.

Finding tight cycles

Let $\mathcal{G}$ be an expander with m edges. So, for every $e \in E(\mathcal{G})$ and $B \subseteq V(\mathcal{G})$ with $|B| \leq \frac{\ell^2}{\varepsilon}$, which is disjoint of e, for at least $(1-\varepsilon)m$ edges $f \in E(\mathcal{G})$: there is a tight path from e to f of length $\leq \ell$ that avoids B.

<u>Theorem (L. 21+)</u>. Let $\mathcal{G}$ be as above. Then $\mathcal{G}$ contains a tight cycle.

<u>Lemma (L.)</u>. In $\mathcal{G}$ as above, for every edge e there is a set $F(e) \subseteq E(\mathcal{G})$ with $|F(e)| \geq (1-\varepsilon)m$, and path $P(e,f)$ of length $\leq \ell$ from e to f, for $f \in F(e)$, s.t. no vertex (except for those in e) appears in $> \frac{\varepsilon m}{\ell}$ paths $P(e,f)$.

Proof of theorem

* For every $e \in E(\mathcal{G})$, let $F(e) \subseteq E(\mathcal{G})$ be a set of size $\geq (1-\varepsilon)m$, and let $P(e,f)$ be a tight path of length $\leq \ell$ from e to f, for $f \in F(e)$, s.t. no vertex appears in $> \frac{\varepsilon m}{\ell}$ paths $P(e,f)$ with $f \in F(e)$.

* For $f \in E(\mathcal{G})$, let $B(f)$ be the set of vertices appearing in $\geq \frac{\varepsilon m}{\ell}$ paths $P(e,f)$.

Then $|B(f)| \cdot \frac{\varepsilon m}{\ell} \leq \ell m \implies |B(f)| \leq \frac{\ell^2}{\varepsilon}$.

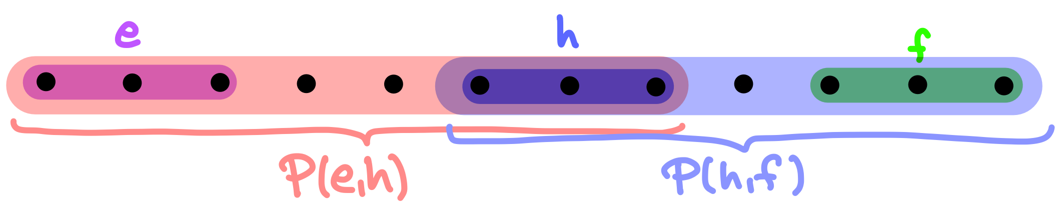
* By expansion, for $\geq (1-\varepsilon)m$ edges e there is a path $Q(f,e)$ of length $\leq \ell$ from f to e that avoids $B(f)$.

Many good pairs

Claim. There are at least $(1-7\varepsilon)m^2$ pairs (e,f) with $e,f \in E(\mathcal{G})$ s.t.:

(a) $Q(f,e)$ is defined,

(b) there are at least $\frac{m}{2}$ edges h s.t. $P(e,h)$ and $P(h,f)$ are defined and the concatenation $P(e,h)P(h,f)$ is a tight path from e to f.



Proof of claim

Let $D(e)$ be the digraph with vertices $E(\mathcal{G})$ where hf is an edge whenever $P(e,h), P(h,f)$ are defined and $P(e,h)P(h,f)$ is a tight path.

For every h s.t. $P(e,h)$ is defined, h has out-degree at least:

$$\underbrace{\#\left(f \in E(\mathcal{G}) : P(h,f) \text{ is defined}\right)}_{\geq (1-\varepsilon)m} - \underbrace{\sum_{\substack{v \in V(P(e,h)) \\ v \notin h}}}_{\leq \ell} \underbrace{\#\left(f : v \text{ is in } P(h,f)\right)}_{\leq \frac{\varepsilon m}{\ell}} \geq (1-2\varepsilon)m.$$

$\Rightarrow$ $D(e)$ has $\geq (1-\varepsilon)m \cdot (1-2\varepsilon)m \geq (1-3\varepsilon)m^2$ edges.

$\Rightarrow$ $D(e)$ has $\geq (1-6\varepsilon)m$ vertices f with in-degree $\geq \frac{m}{2}$.

$\Rightarrow$ (b) is satisfied for $\geq (1-6\varepsilon)m^2$ pairs (e,f).

End of proof of theorem

Let $e, f \in E(\mathcal{G})$ satisfy:

(a) $Q(f,e)$ is defined,

(b) there are at least $\frac{m}{2}$ edges h s.t. $P(e,h), P(h,f)$ are defined and the concatenation $P(e,h)P(h,f)$ is a tight path from e to f.

* There are at most $\frac{\varepsilon m}{\ell} \cdot \ell = \varepsilon m$ edges h as in (b) s.t. $P(e,h)$ and $Q(f,e)$ have a vertex in common (which is not in e).

* There are at most $\frac{\varepsilon m}{\ell} \cdot \ell = \varepsilon m$ edges h as in (b) s.t. $P(h,f)$ and $Q(f,e)$ have a vertex in common (which is not in f).

$\Rightarrow$ There are $\geq \frac{m}{2} - 2\varepsilon m \geq \frac{m}{4}$ edges h as in (b) s.t. $P(e,h)P(h,f)Q(f,e)$ is a tight cycle.

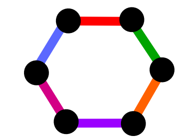
Other applications

The same arguments yield the following results (already proved by O. Janzer '20).

* Every properly edge-coloured graph on n vertices with $\geq cn(\log n)^5$ edges contains a rainbow cycle.

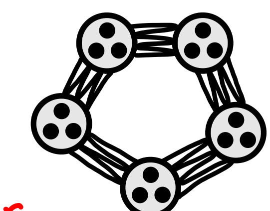
  incident edges have distinct colours

  all edges have distinct colours



  (using tools from Jiang-Methuku-Yepremyan '21; observed by JLMY '21+).

* Every graph on n vertices with $\geq cn^{2-1/r}(\log n)^{5/r}$ edges contains an r-blowup of a cycle.



  vertices replaced by independent sets of size r, edges replaced by $K_{r,r}$

  (using a result by Morris-Saxton '16; connection observed by JLMY).

Open problems

* What is $f_r(n)$ = max number of edges in an r-graph on n vertices with no tight cycles? Best known bounds: $c_1 n^{r-1}\frac{\log n}{\log\log n} \leq f_r(n) \leq c_2 n^{r-1}(\log n)^5$.

  B. Janzer '20   L. 21+

* <u>Question (Conlon '11)</u>. Is there $c=c(r)$ s.t. if $\ell > r$ and $r \mid \ell$ then r-graphs on n vertices with no tight cycles of length ℓ have $O(n^{r-1+\frac{c}{\ell}})$ edges?

  (Every r-partite r-graph has no tight cycles of length ℓ with $r \nmid \ell$.)

* What is $g(n)$ = max number of edges in a properly edge-coloured graph on n vertices with no rainbow cycles?

  Best known bounds: $c_1 n \log n \leq g(n) \leq c_2 n (\log n)^4$.

  Keevash–Mubayi–Sudakov–Verstraëte '07   O. Janzer '20

Thank you!