Finding monotone patterns

Shoham Letzter

University College London

joint with Omri Ben-Eliezer and Erik Waingarten

ICALP

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Property testing

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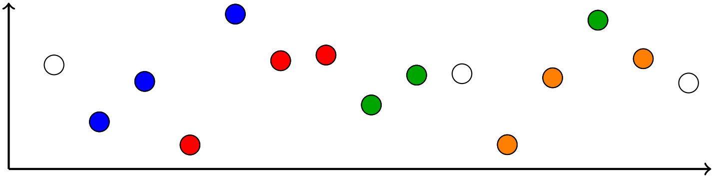
We consider testing with **one-sided error**: given an object which is far from having $\mathcal{P}$, provide evidence of not being in $\mathcal{P}$, with high probability.

Testing for $(1 \dots k)$-freeness

Fix $k \geq 2$.

- **Input.** $f : [n] \to \mathbb{R}$ with $\Omega(n)$ disjoint increasing k-tuples.



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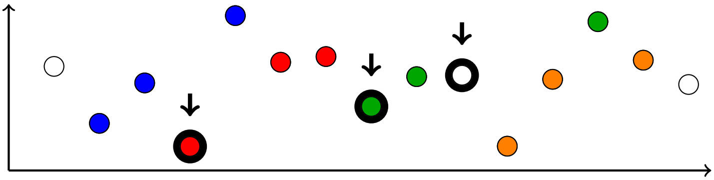

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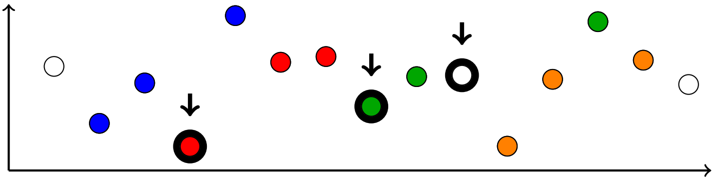

We sometimes refer to an increasing k-tuple as a **$(1 \ldots k)$-copy**.

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Newman–Rabinovich–Rajendraprasad–Sohler '17. For $k \geq 2$, there is a (non-adaptive) tester which makes $(\log n)^{O(k^2)}$ queries.

Our results

Theorem (Ben-Eliezer–Canonne–L.–Waingarten)

An optimal non-adaptive algorithm for testing $(1\dots k)$-freeness makes $\Theta_k\left((\log n)^{\lfloor \log_2 k \rfloor}\right)$ queries.

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Theorem (Ben-Eliezer–L.–Waingarten)

An optimal adaptive algorithm for testing $(1\ldots k)$-freeness makes $\Theta_k(\log n)$ queries.

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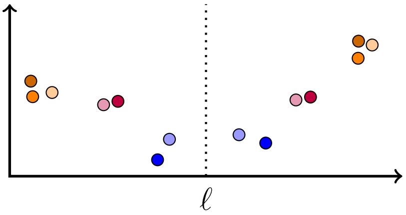

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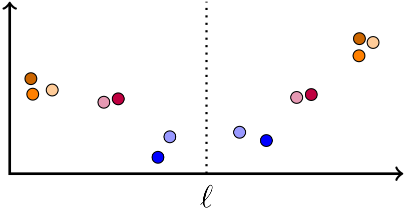

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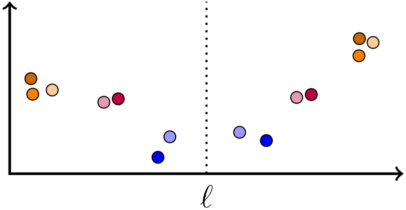

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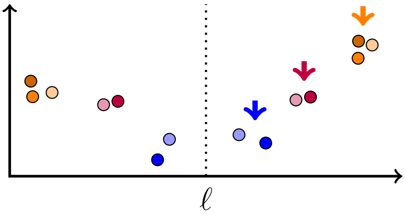

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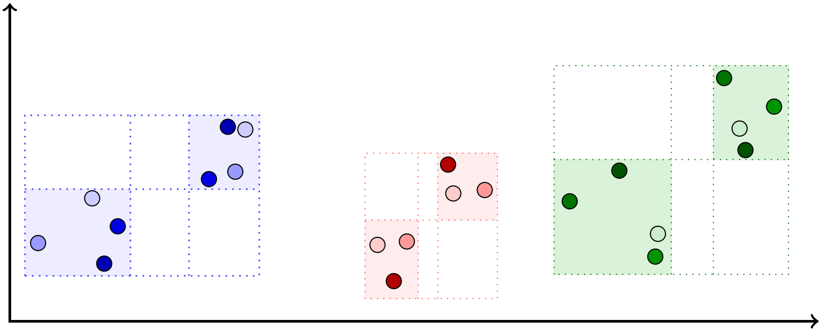

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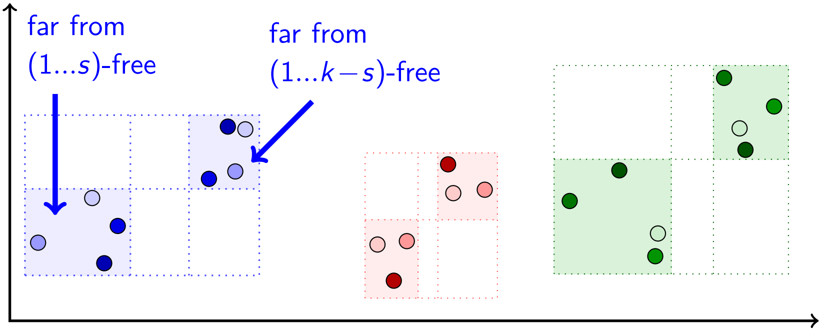

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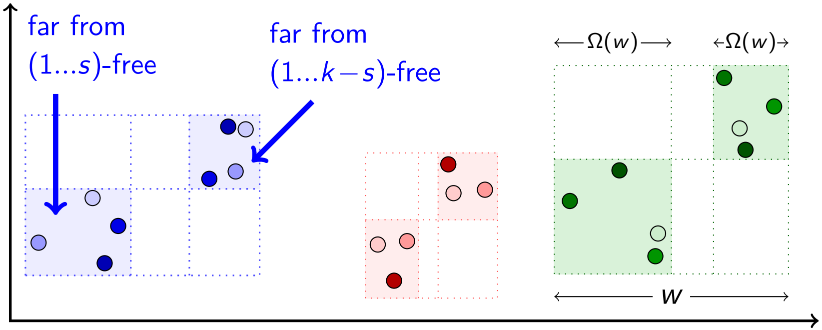

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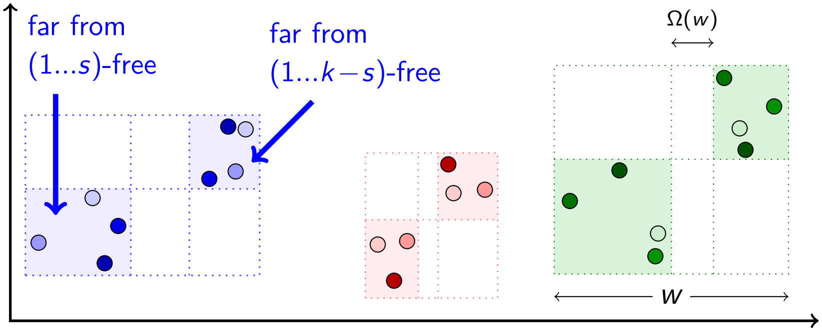

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Thus, if f is structured, there is a **robust collection** of disjoint 'splittable intervals' $\mathcal{J}$ s.t.

- $\sum_{J\in\mathcal{J}} |J| = \Omega(n)$,
- if K contains $J \in \mathcal{J}$ then K has $\Omega(|K|)$ disjoint increasing k-tuples.

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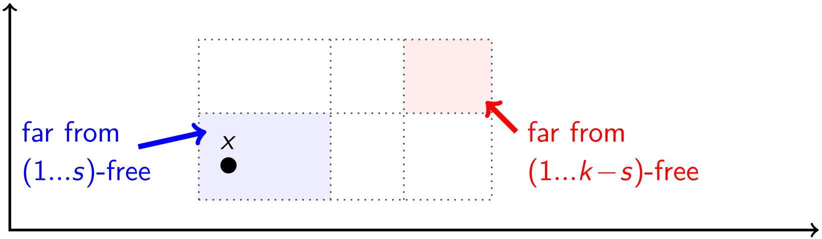
- $\Theta(1)$ queries to find x in bottom-left of robust splittable interval I.



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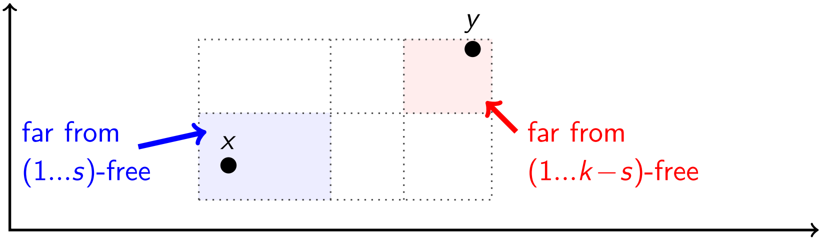

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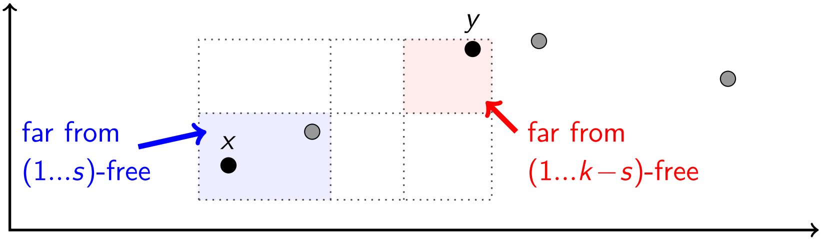

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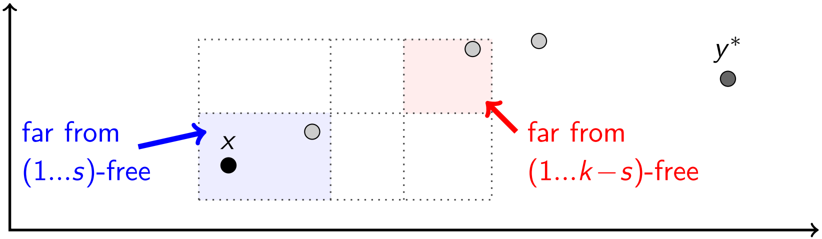

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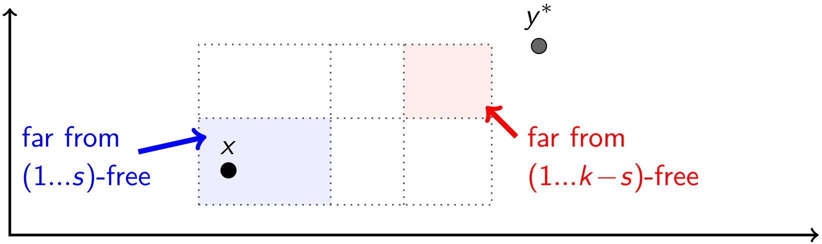

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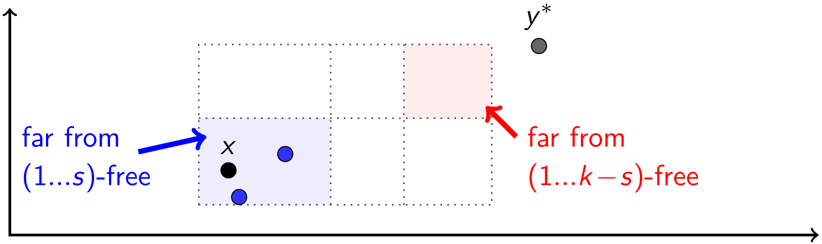

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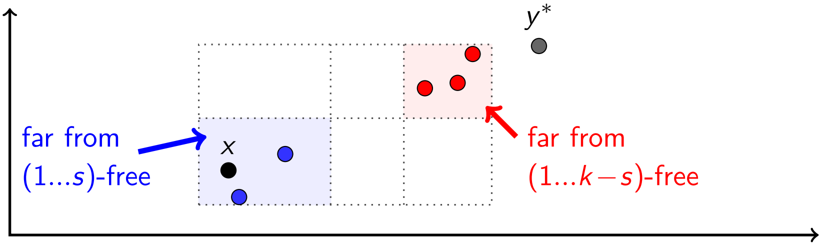

Handling overshooting in the structured case – I

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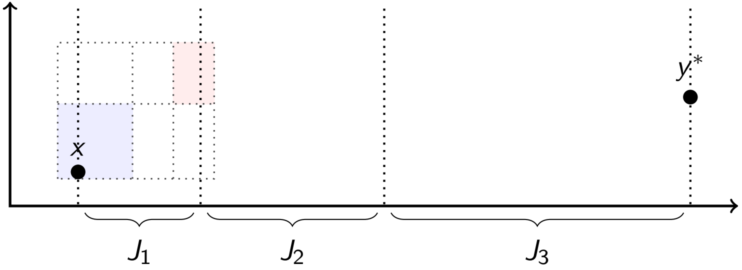

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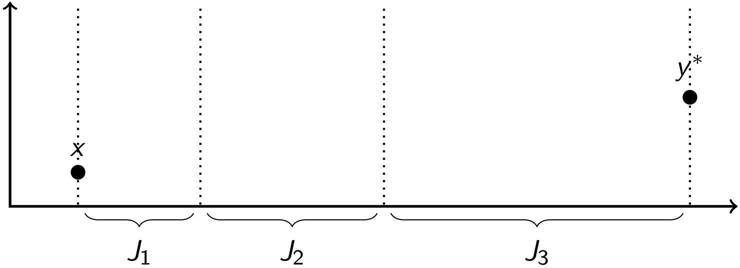

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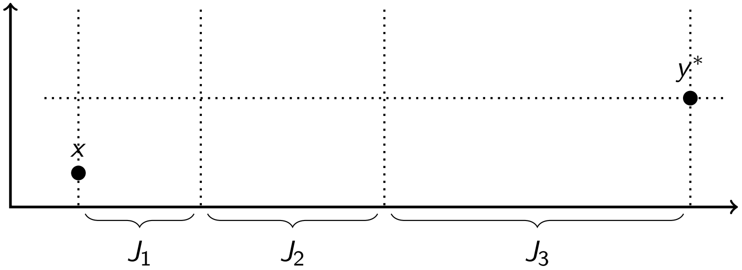

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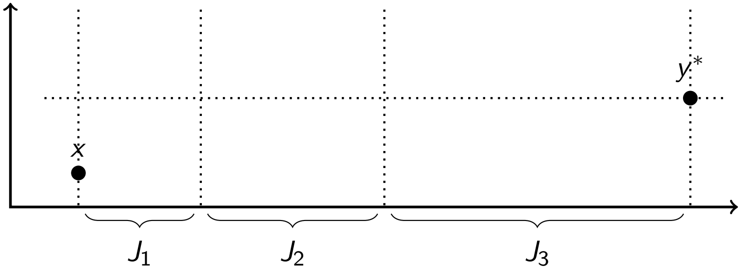

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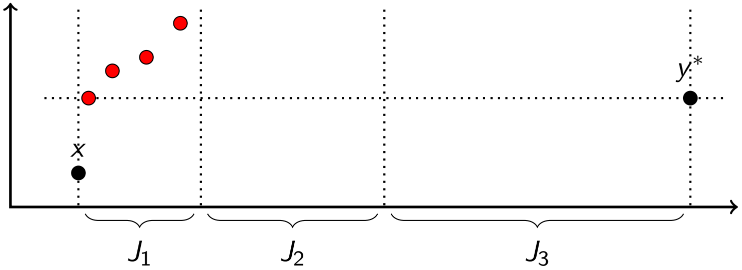

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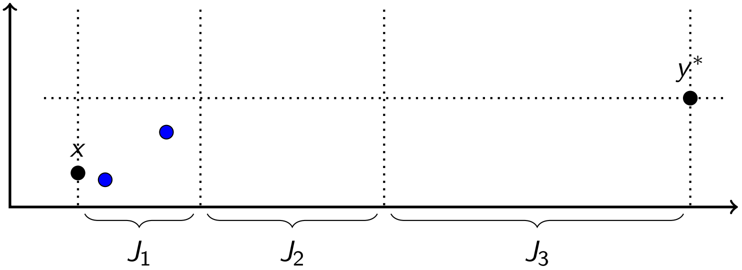

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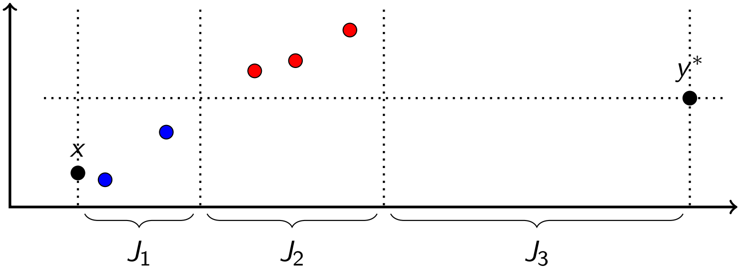

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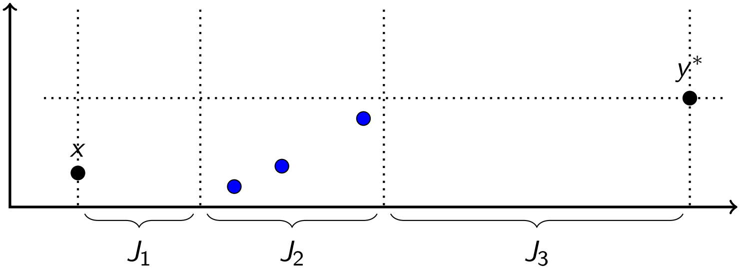

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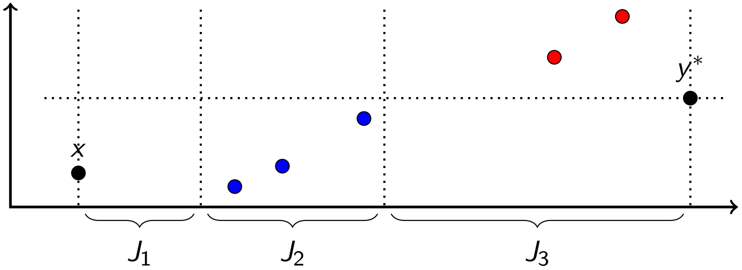

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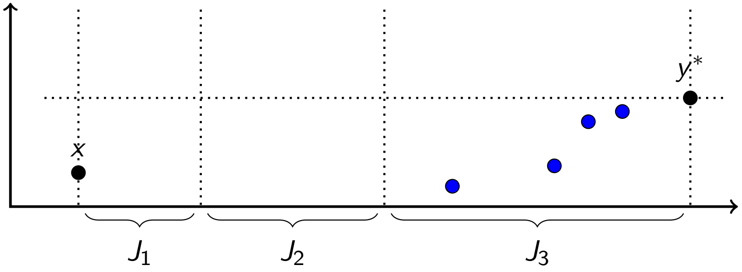

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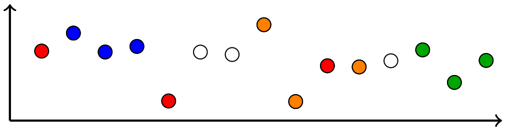

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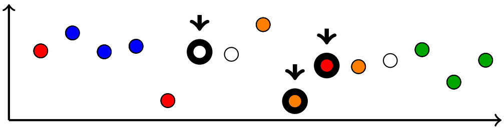

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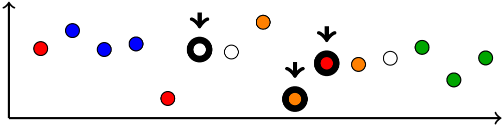

Newman–Varma '21: adaptive π-freeness tester with $n^{o(1)}$ queries.

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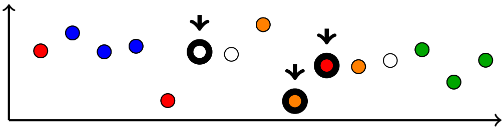

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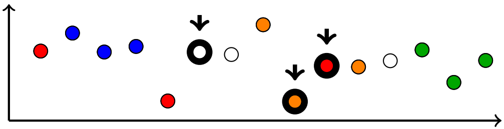

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- Finding a π-copy (length k) in a permutation of length n:

Fox, '13. $2^{O(k^2)}n$.

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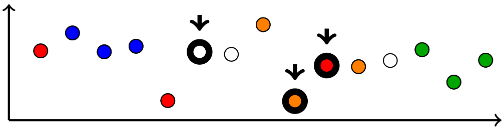

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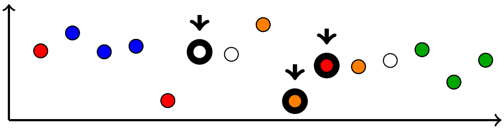

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Thank you!!!