χ-boundedness of graphs with no cycle with k chords

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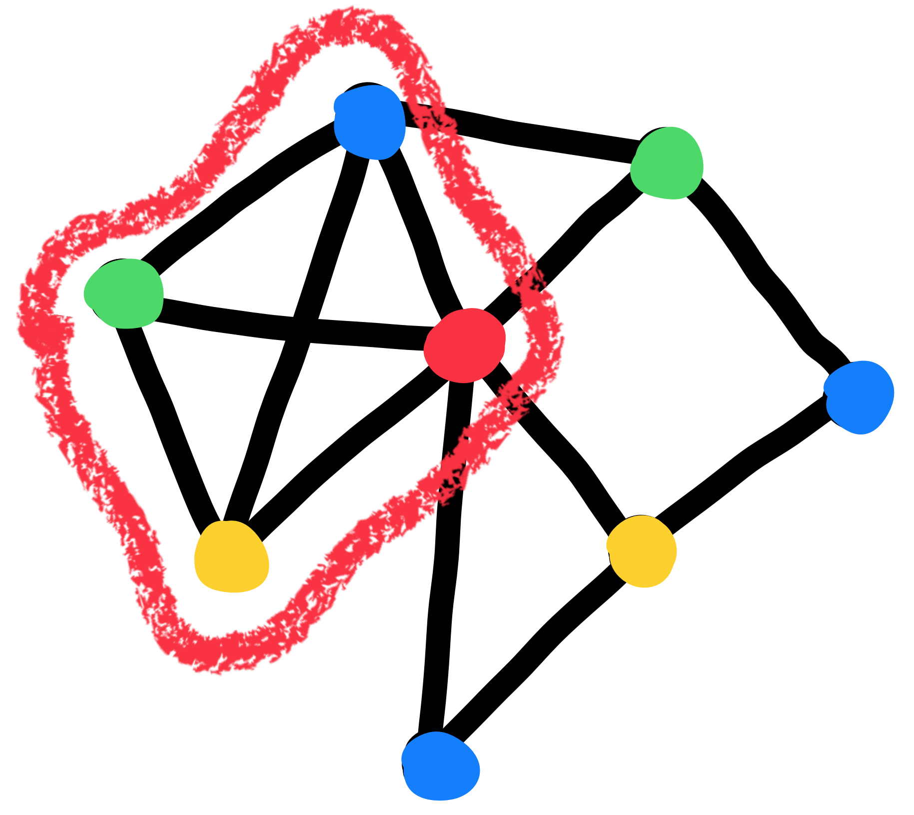
$\omega(G)$ = size of largest clique in G.

$\chi(G)$ = min number of colours in a $\underbrace{\text{proper colouring}}_{\text{adjacent vertices have different colours}}$ of $V(G)$.



Observation: $\chi(G) \geq \omega(G)$ for every graph G.

The opposite is far from true. E.g.: there exist Δ-free graphs with arbitrarily large χ.

Chudnovsky–Robertson–Seymour–Thomas '06: $\chi(H) = \omega(H)$ for every induced subgraph of G iff neither G nor G^c has induced odd cycle of length ≥ 5.

Gyárfás '87: A $\underbrace{\text{class of graphs}}_{\text{closed under taking induced subgraphs.}}$ $\mathcal{G}$ is **χ-bounded** if there is a function f s.t. $\forall G \in \mathcal{G}$ $\chi(G) \leq f(\omega(G))$.

Gyárfás '87: A class $\mathcal{G}$ is **χ-bounded** if there is f s.t. $\forall G \in \mathcal{G}$ $\chi(G) \leq f(\omega(G))$.

Forb(H) = {graphs with no induced copy of H}.

- If H contains a cycle then Forb(H) is not χ-bounded.

  Erdős '59: $\exists$ graphs with arbitrarily large χ and girth.

  length of shortest cycle

- Forb(S_k) is χ-bounded $\forall k$.

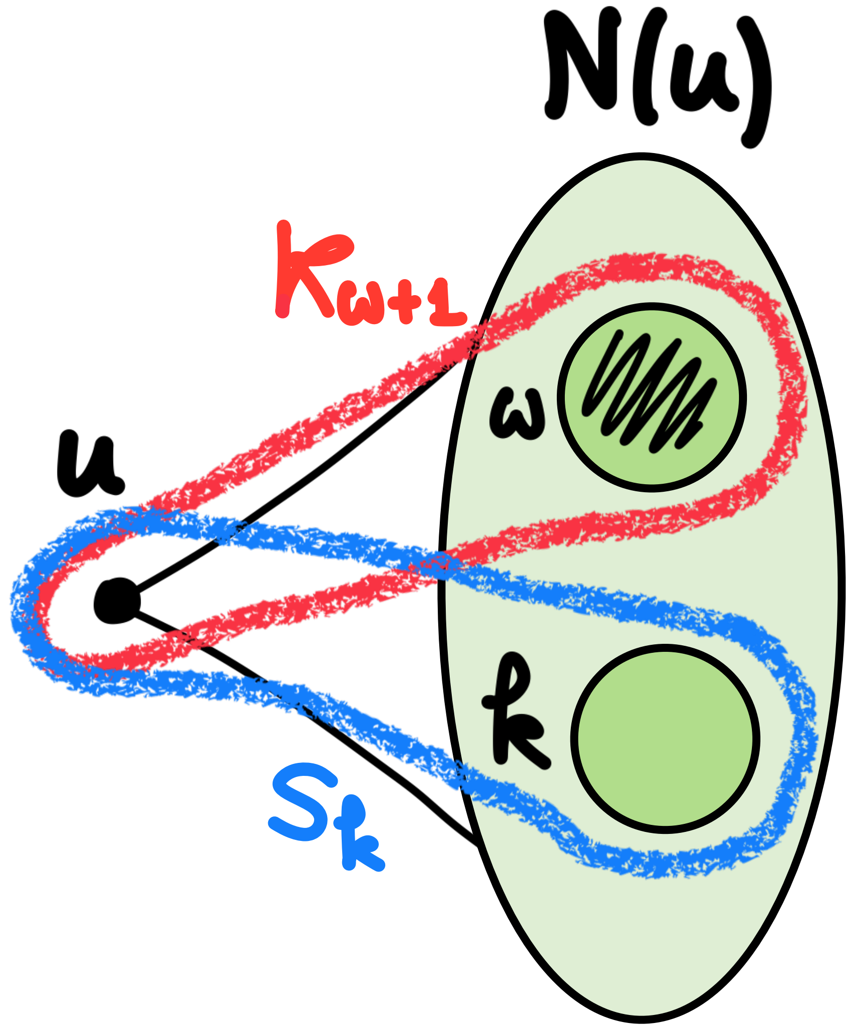
  star on k edges

  Let $G \in \text{Forb}(S_k)$, $u \in V(G)$.

  $N(u)$ has no clique of size $\omega(G)$ or independent set of size k.

  $|N(u)| \leq r(K_{\omega(G)}, K_k)$ $\forall u \Rightarrow \chi(G) \leq r(K_{\omega(G)}, K_k) + 1$.

  Ramsey number



- **Gyárfás '87:** Forb(P_k) is χ-bounded $\forall k$.

  path on k vertices

Forbidding a tree

Forb(H) • is χ-bounded if H is a path or star and
• isn't χ-bounded if H contains a cycle.

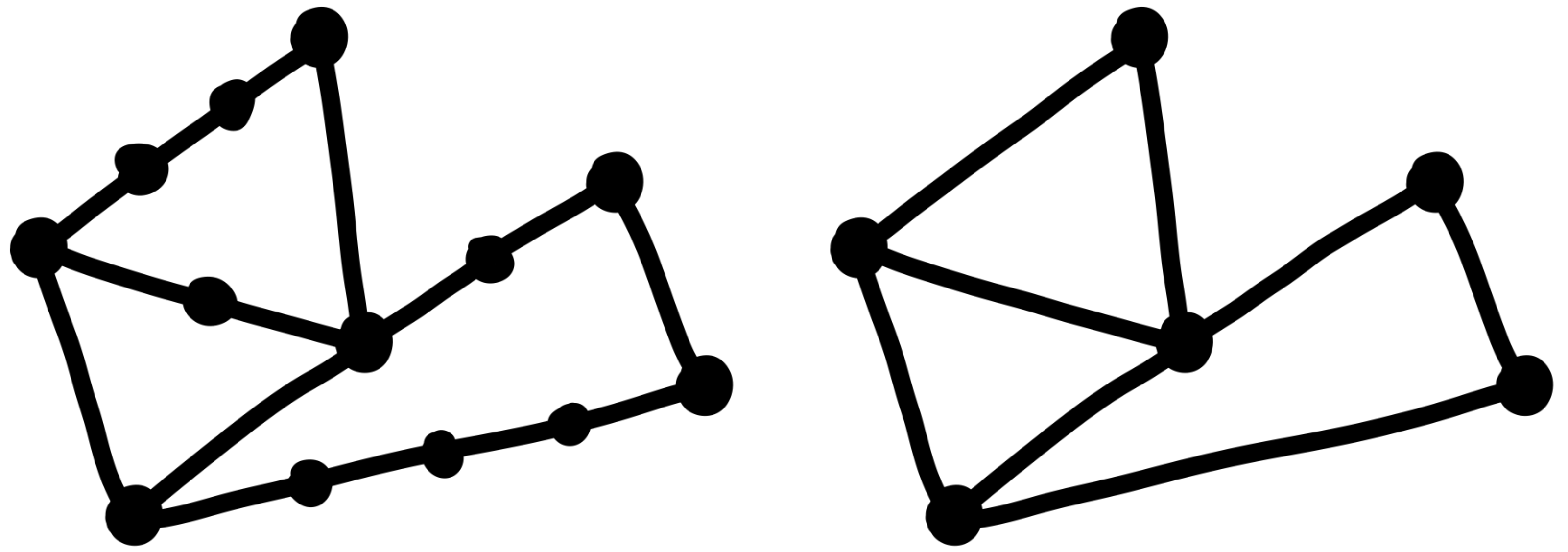
Conjecture (Gyárfás '87): Forb(H) is χ-bounded iff H is a forest.
($\Leftrightarrow$ Forb(T) is χ-bounded $\forall$ tree T.)

- **Kierstead-Penrice '94:** Forb(T) is χ-bounded $\forall$ tree T of radius ≤ 2.
- **Kierstead-Zhu '06:** Forb(T) is χ-bounded $\forall$ tree T of radius ≤ 3.

$\text{Forb}^*(H)$ = {graphs with no induced copy of a subdivision of H.

- **Scott '97:** $\text{Forb}^*(T)$ is χ-bounded $\forall$ tree T.

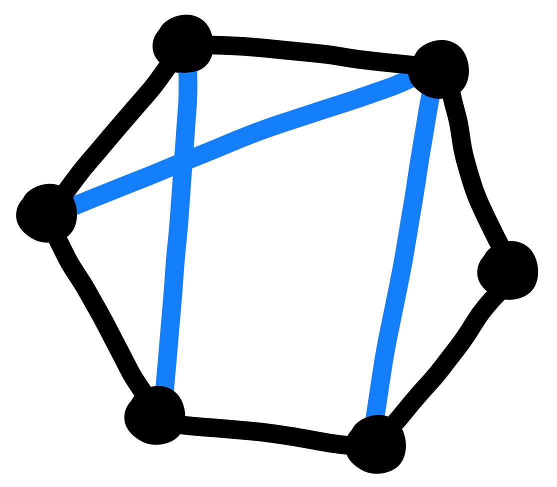
Question: For which H is $\text{Forb}^*(H)$ χ-bounded?



$\mathcal{C}_k$ = {graphs with no cycle with exactly k chords}.

Conjecture (Aboulker–Bousquet '15): $\mathcal{C}_k$ is χ-bounded $\forall k$.

- **Trotignon–Vušković '10:** $\mathcal{C}_1$ is χ-bounded.
- **Aboulker–Bousquet '15:** $\mathcal{C}_2$ and $\mathcal{C}_3$ are χ-bounded.



a cycle with 3 chords

Theorem (Lee–L.–Pokrovskiy '21+):

There exists k_0 s.t. $\mathcal{C}_k$ is χ-bounded $\forall k \geq k_0$.

Need to show: if $\chi(G) \gg \omega(G), k$ then G has a cycle with exactly k chords.

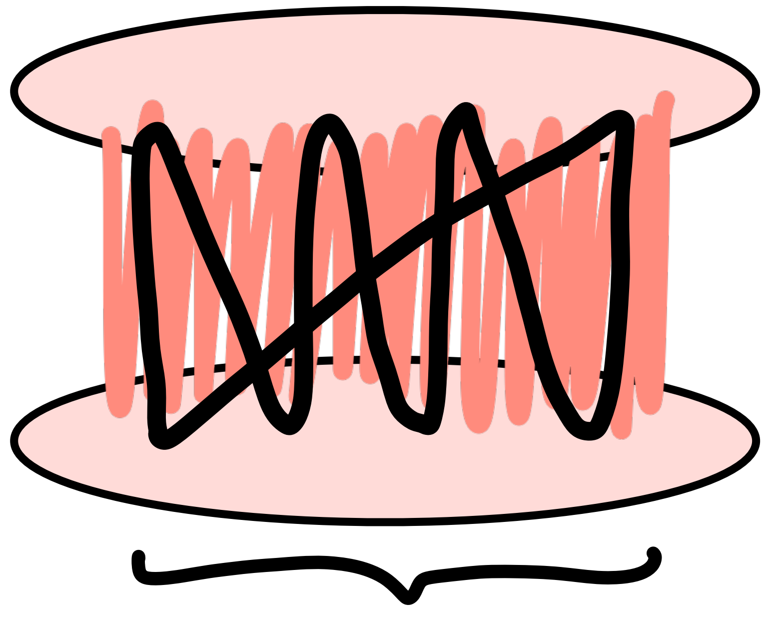
Theorem: $\forall k,\ell$ $\exists f$ s.t. if $\chi(G) \geq f(\omega(G))$ then G contains an induced $K_{\ell,\ell}$ or a cycle with k chords.

(Uses **Bousquet-Thomasse '15:** $\text{Forb}(\Delta, K_{\ell,\ell}, W_k)$ is χ-bounded $\forall k,\ell$.)

W_k: k-wheel

A cycle C of length $2a$ in a $K_{\ell,\ell}$ has $a^2 - 2a$ chords.

a^2: #edges induced by $V(C)$; $2a$: edges in C



$\Rightarrow$ If $k = a^2 - 2a$ for some integer a then $\exists$ cycle with k chords.

Corollary: $\forall k,\ell$ $\exists f$ s.t. if $\chi(G) \geq f(\omega(G))$ then G contains ℓ disjoint induced $K_{\ell,\ell}$'s with no edges between them or a cycle with k chords.

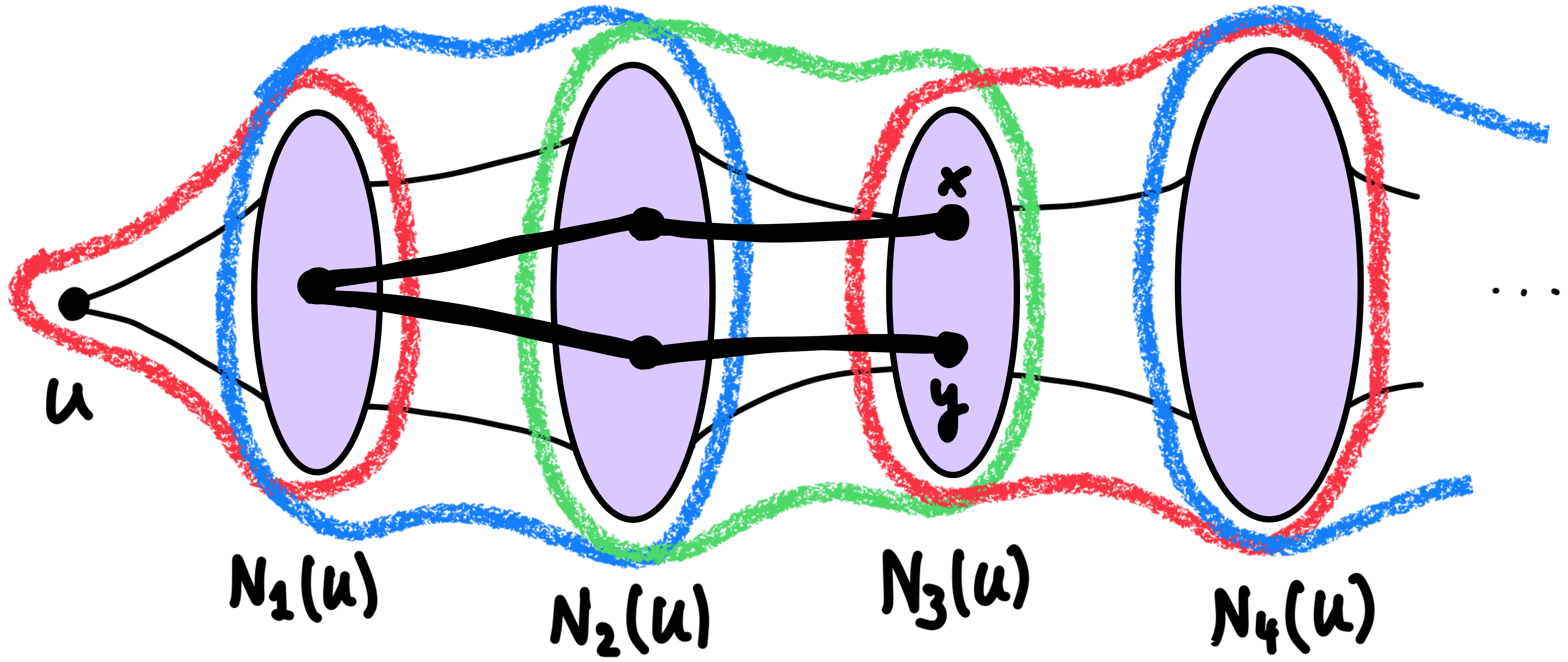
Unimodal paths

Fix $u \in V(G)$. $N_i(u) = \{v : \mathrm{dist}(u,v) = i\}$.

$\chi(G) \le \max_i \{\chi(N_i(u) \cup N_{i+1}(u))\} \le \max_i \{\chi(N_i(u)) + \chi(N_{i+1}(u))\}$

$\Rightarrow \exists i$ s.t. $\chi(N_i(u)) \ge \frac{1}{2} \cdot \chi(G)$. Write $G_1 = G[N_1(u)]$.

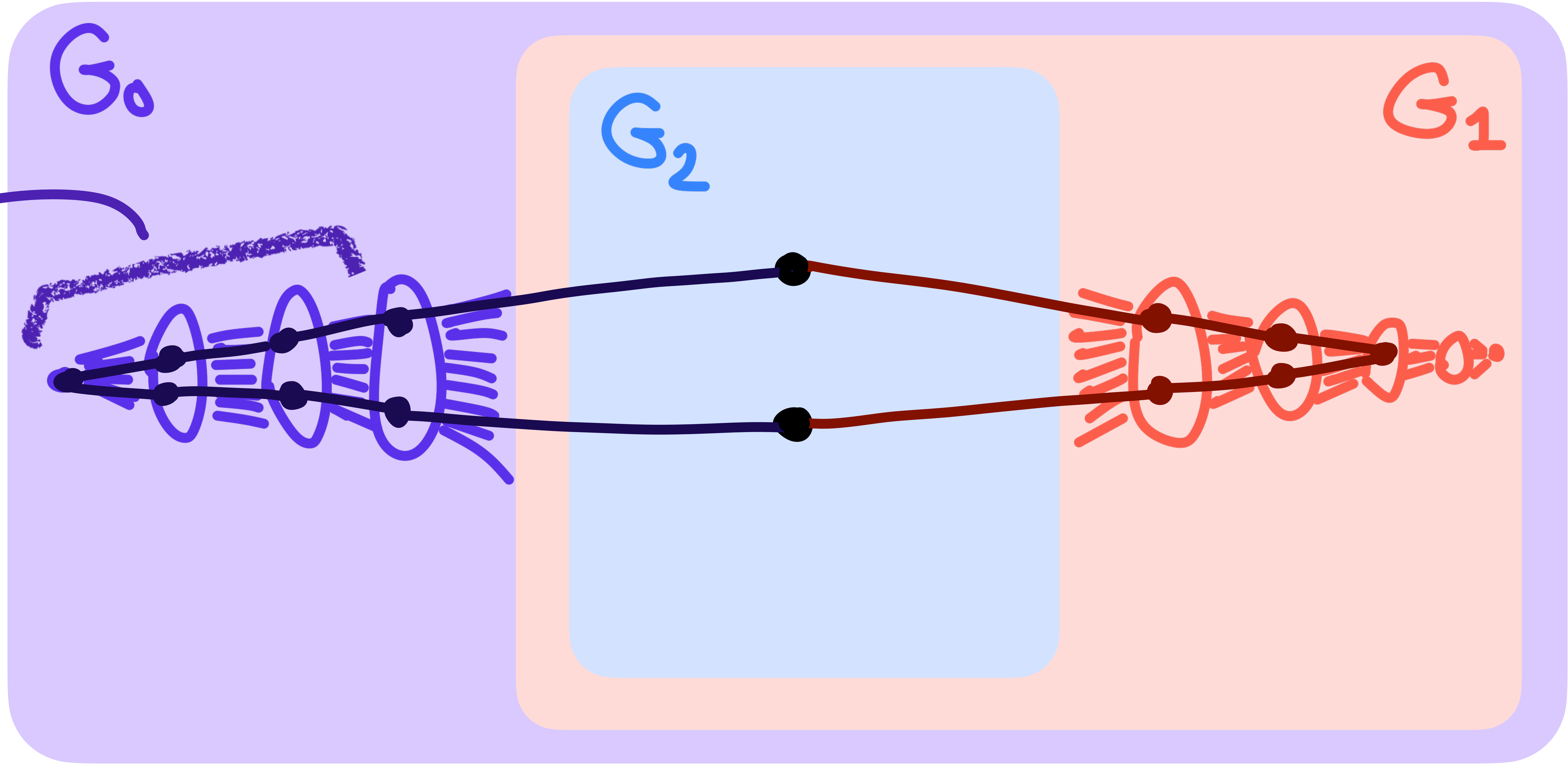
Observation: $\forall x,y \in V(G_1)$ $\exists$ induced path from x to y whose interior is in $V(G) \setminus V(G_1)$. Such a path is called **unimodal**.



Repeat the process to obtain $G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_p$ s.t.:

- $\chi(G_p) \geq 2^{-p}\chi(G)$
- $\forall x, y \in V(G_p)$ $\exists$ unimodal path from x to y with interior in $V(G_i) \setminus V(G_p)$.

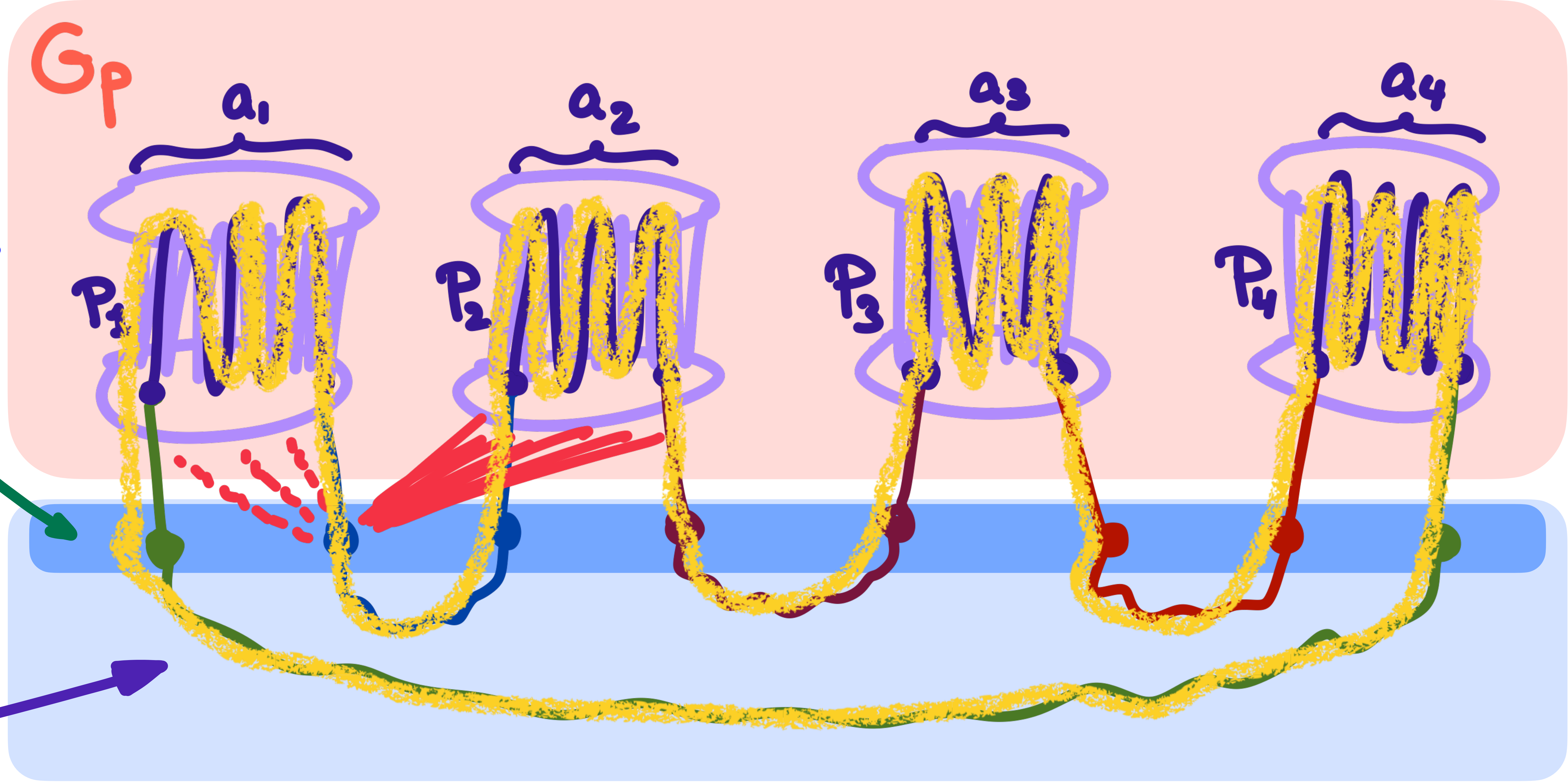
no edges to the unimodal path in G_1



$K_1, \ldots, K_{100}$: disjoint $K_{\ell,\ell}$'s in G_p with no edges between them.

Take P_i a path of length $2a_i$ in K_i.

Join by unimodal paths from distinct G_js.



May assume: $\forall$ part X of K_i and u here
u is adjacent to all or none of X.

Lemma: $\exists\, O(\sqrt{k})$ chords here

$\rightsquigarrow$ #chords in depicted cycle $= \sum_{i=1}^{100} \left(a_i^2 + \overset{O(1)}{\widetilde{\sigma_i}}\, a_i\right) + \overbrace{O(\sqrt{k})}^{\text{independent of } a_i\text{'s}}.$

$\forall a_1, \ldots, a_{100}$ $\exists$ cycle with the following number of chords: $\sum_{i=1}^{100} (a_i^2 + \sigma_i a_i) + O(\sqrt{k})$.

Lemma. $\forall c$, large k: $\exists a_1, \ldots, a_{20} \geq c$ s.t. $k = a_1^2 + \ldots + a_{20}^2$.

Lemma. $\forall c$, large k s.t. $4 | k$: $\exists a_1, \ldots, a_{80} \geq c$ s.t. $k = (a_1^2 + a_1) + \ldots + (a_{80}^2 + a_{80})$.

Apply one of above lemmas (depending on parity of σ_i) to find a cycle with k' chords, where $k' \in \{k, k-1, k-2, k-3\}$.

To get exactly k chords we work much harder to find gadgets to offset parity.

graphs with no cycle with k chords

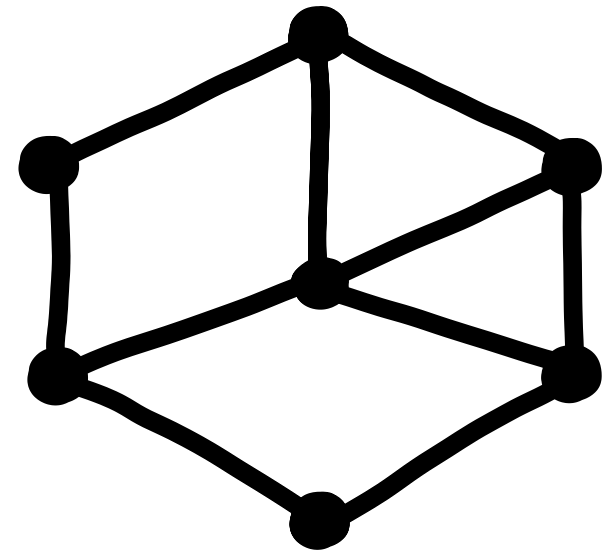
Conjecture (Aboulker–Bousquet '15): $\mathcal{C}_k$ is χ-bounded $\forall k$.

Known for $k=1,2,3$; we proved for large k and $k=a^2-2a$, $a\in\mathbb{N}$.

A *wheel* is an induced cycle C plus a vertex adjacent to ≥ 3 vertices of C.

Conjecture (Trotignon '13): {wheel-free graphs} is χ-bounded.

Aboulker–Bousquet '15: {wheel & triangle-free} is χ-bounded.



Conjecture (Gyárfás '87): Forb(T) is χ-bounded $\forall$ tree T.

Question: when is Forb$^*(H)$ χ-bounded?

Thank you for your attention!