Monochromatic directed paths in random tournaments

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joint work with Matija Bucić and Benny Sudakov

ETH–ITS

Random Structures and Algorithms

August 2017

Ramsey theory for directed graphs

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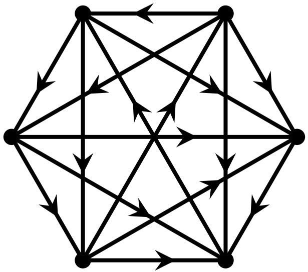

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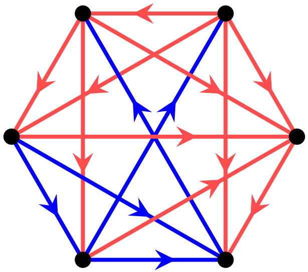

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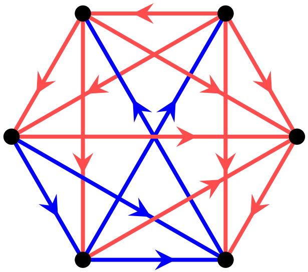

Basic question.

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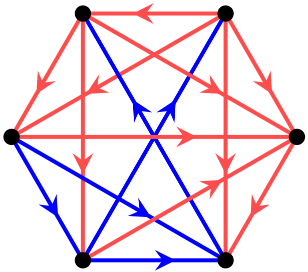

Basic question. which digraphs appear as monochromatic subgraphs of every 2-coloured tournament of order n?

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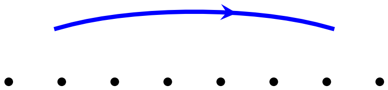

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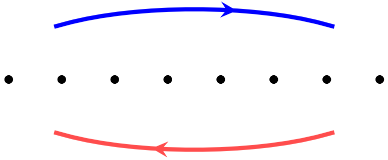

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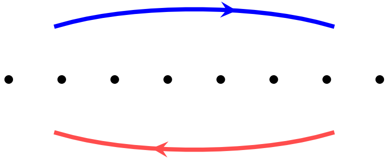

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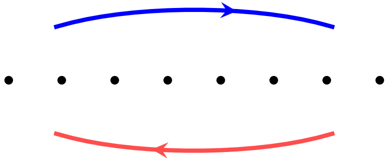
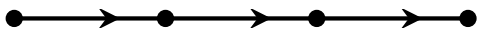


We focus on **directed paths** ().

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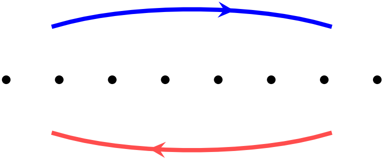
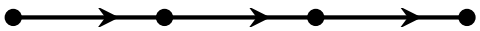


We focus on **directed paths** ().
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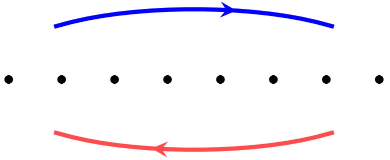
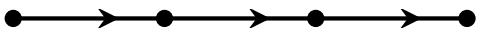


We focus on **directed paths** ().
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Definition

$l(T) = \max \left\{ l : \textit{every 2-colouring of } T \textit{ has a monochromatic } \overrightarrow{P_l} \right\}.$

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- By GHRV theorem, there is a monochromatic $\overrightarrow{P_{\sqrt{n}}}$. □

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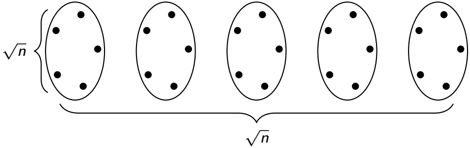

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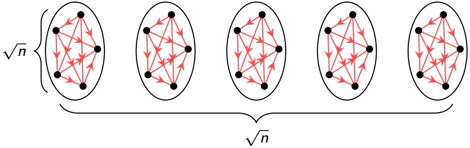

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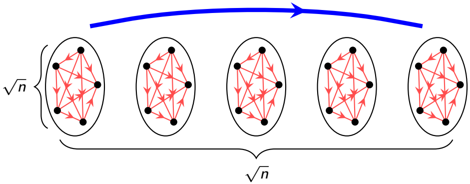

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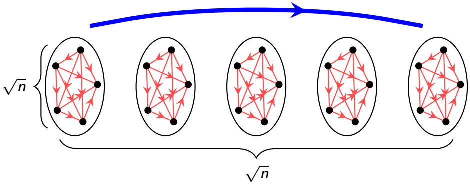

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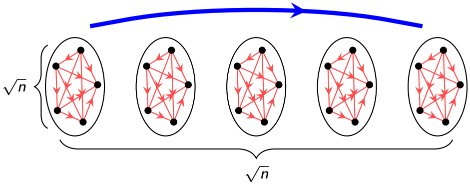

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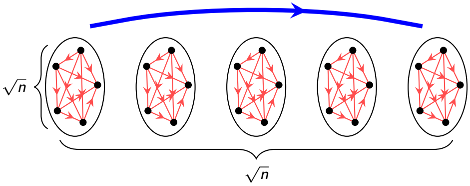

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Question

What is $\max\{I(T) : T$ *is a tournament on* n *vertices*$\}$*?*

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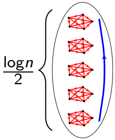

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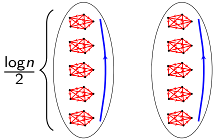

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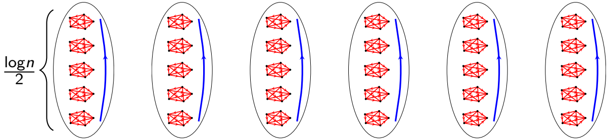

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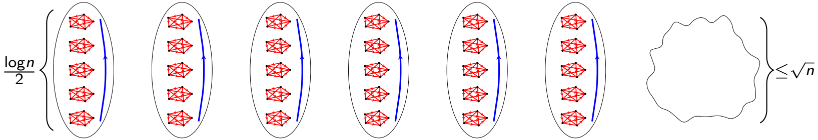

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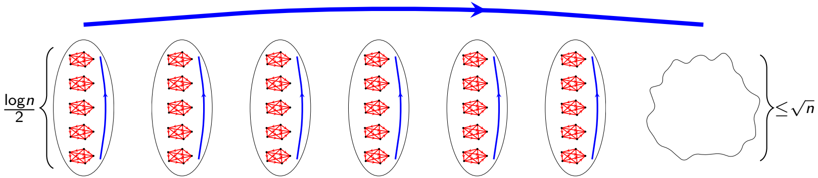

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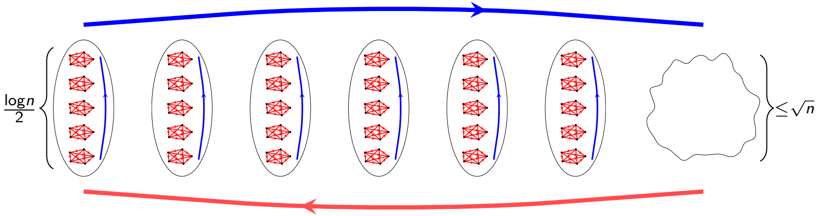

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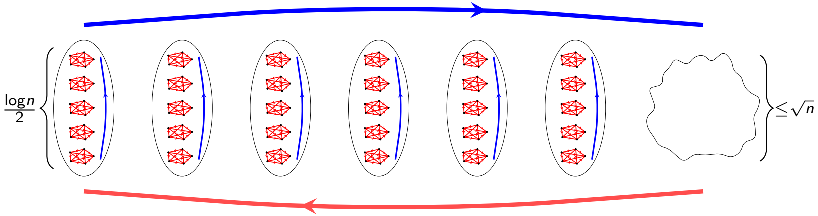

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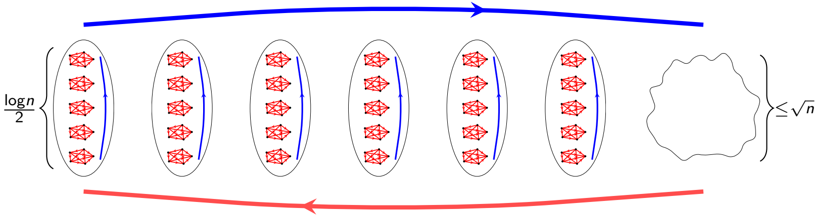

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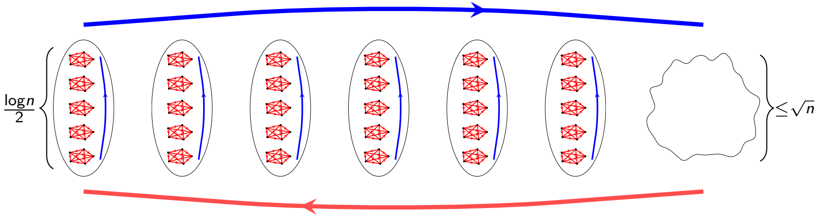

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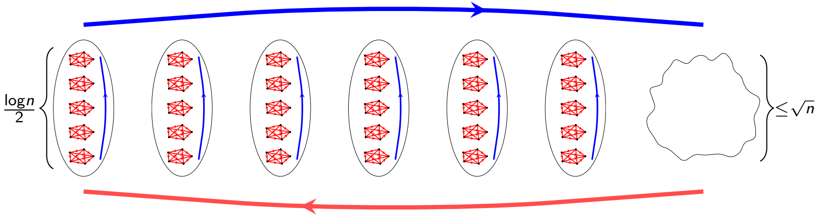

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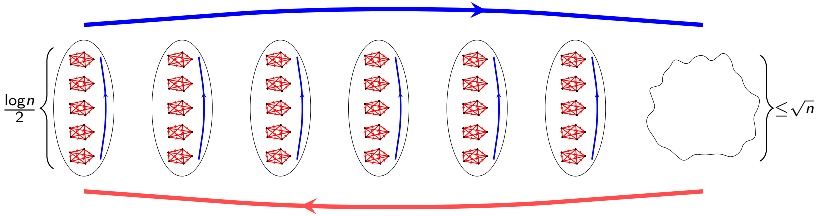

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Note: H is a 2-colouring of the complete directed graph on k vertices.

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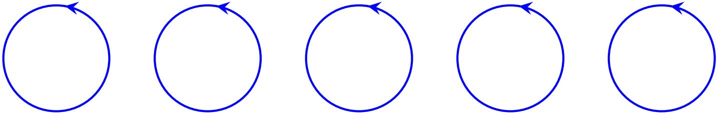

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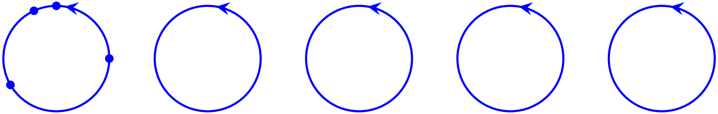

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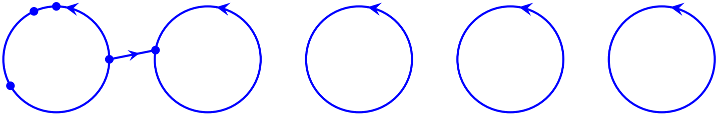
Suppose that H has a blue $\overrightarrow{P_{k/2}}$.

We find a blue path of length $n/200$ in T:



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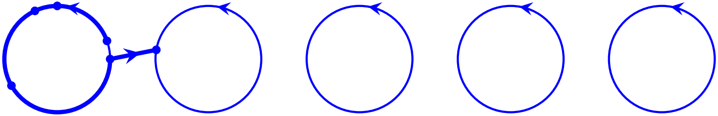

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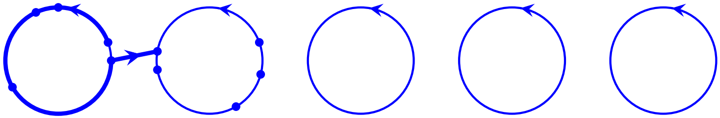

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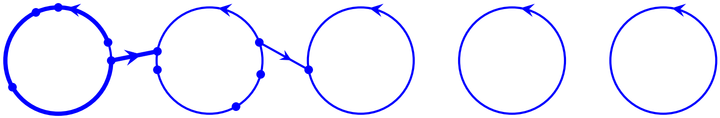

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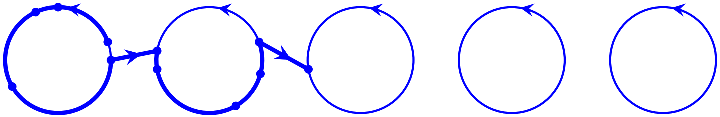

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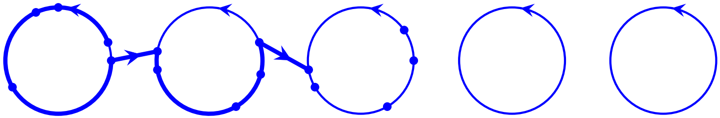

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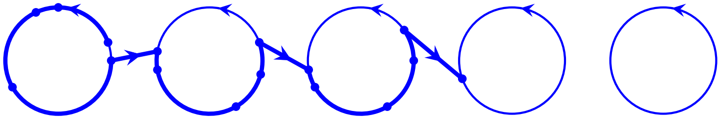

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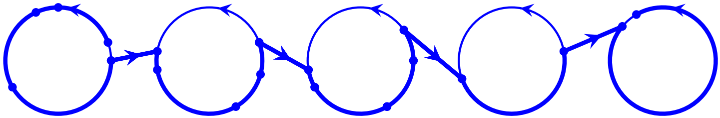

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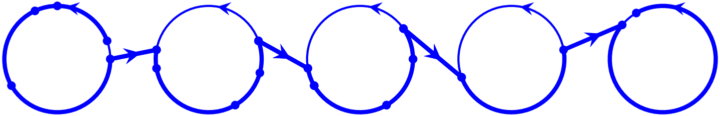

O.w., there is a matching of $k/4$ two-sided red edges ().

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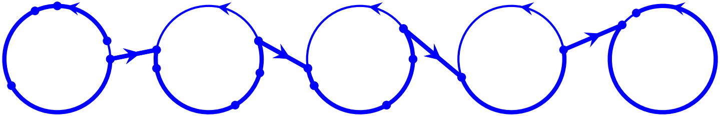

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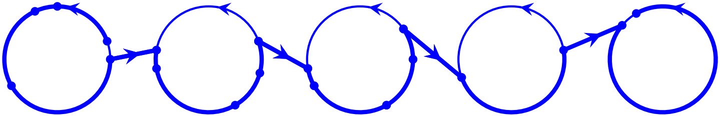

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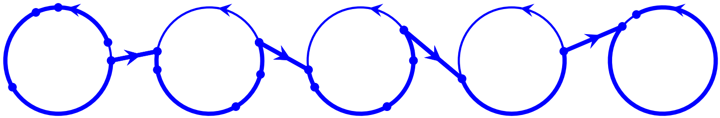

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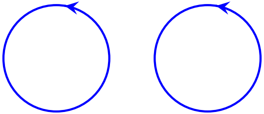

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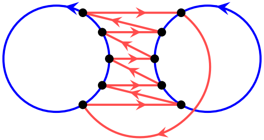

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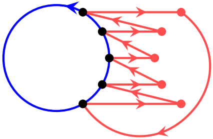

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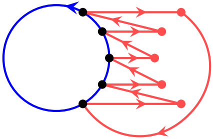

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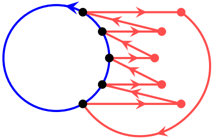

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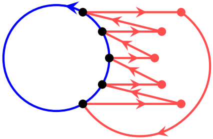

So, there are vertex-disjoint blue medium cycles $C'_1, \dots, C'_{k/4}$ and vertex-disjoint red medium cycles $C''_1, \dots, C''_{k/4}$, such that $|V(C'_i) \cap V(C''_i)| \geq \gamma \log n$.

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$C'_1, \ldots, C'_{k/4}$ vertex-disjoint blue medium cycles; $C''_1, \ldots, C''_{k/4}$ vertex-disjoint red medium cycles; $|V(C'_i) \cap V(C''_i)| \geq \gamma \log n$.

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Hence, there is a monochromatic $\overrightarrow{P_{k/8}}$.

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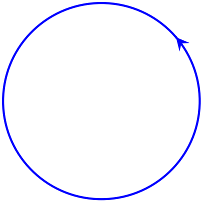

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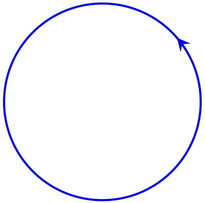

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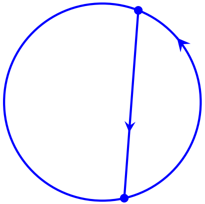

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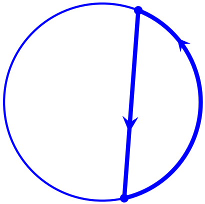

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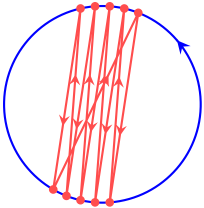

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We find a medium red cycle, a contradiction.

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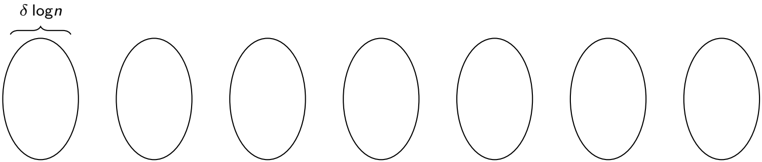

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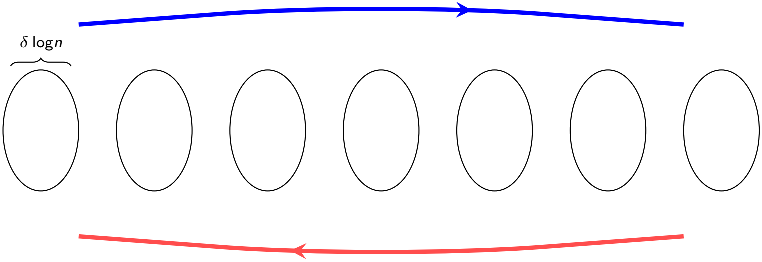

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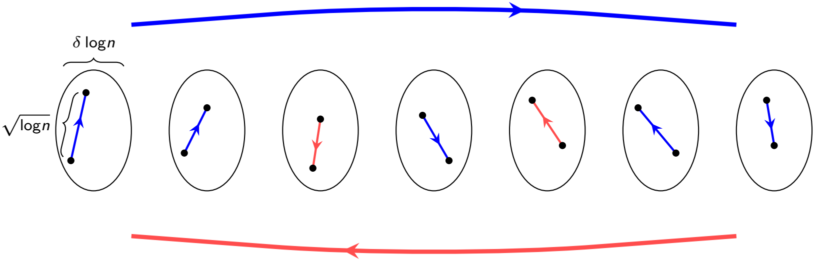

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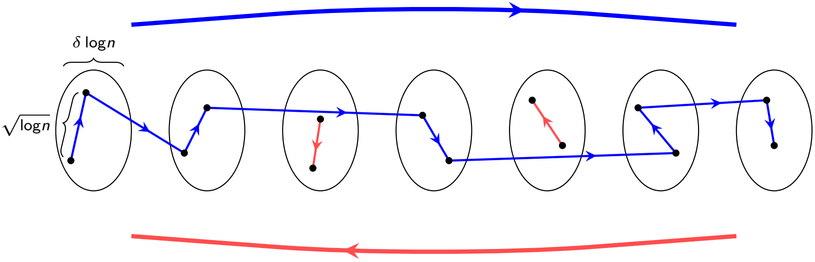

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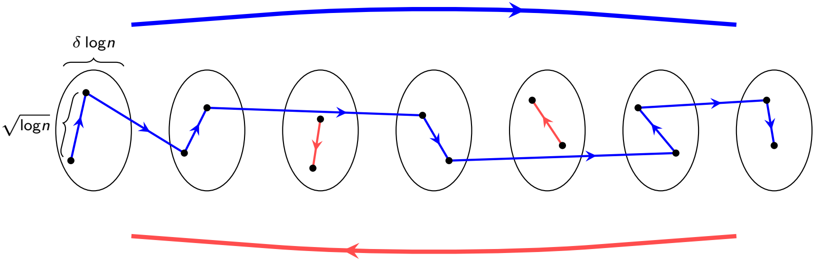

We found a monochromatic path of length $\Omega\left(\frac{n}{\sqrt{\log n}}\right)$. □

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Somewhat surprisingly, Beck ('83) showed: $r_e(P_n) = O(n)$.

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The end

Thank you for listening!