Directed Ramsey theory

Shoham Letzter

joint work with Matija Bucić and Benny Sudakov

ETH - ITS

Colloquia in Combinatorics - LSE

May 2017

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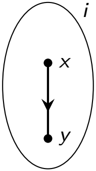

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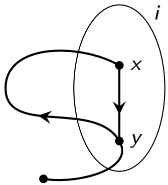

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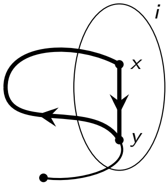

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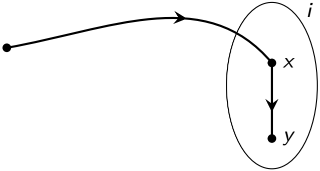

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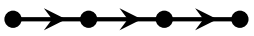
Denote by $\overrightarrow{P_k}$ the directed path of order k ($P_4 =$ ).

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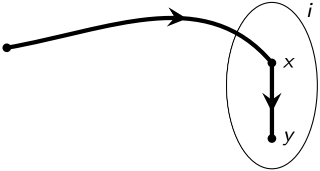

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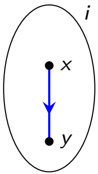

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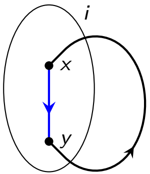

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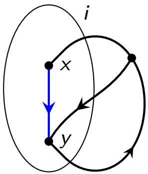

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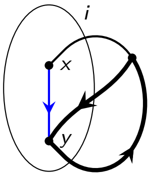

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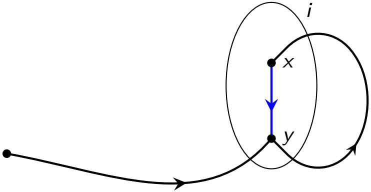

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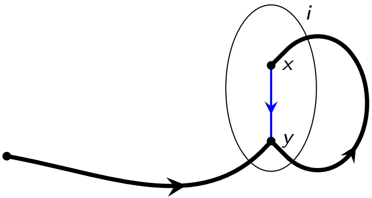

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What about oriented trees?

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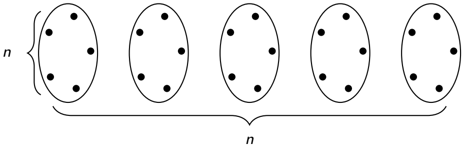

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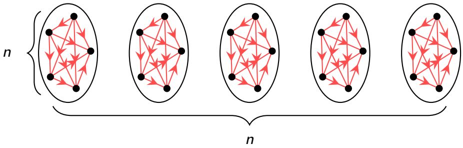

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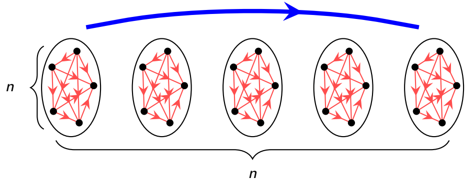

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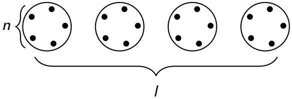

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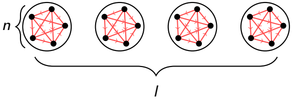

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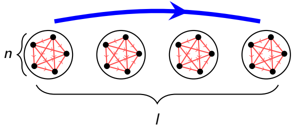

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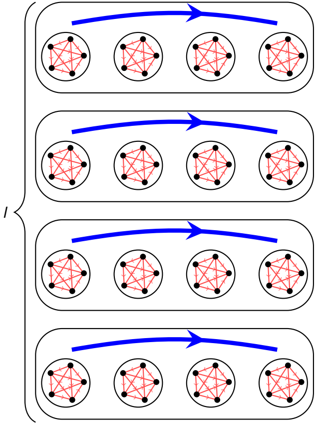

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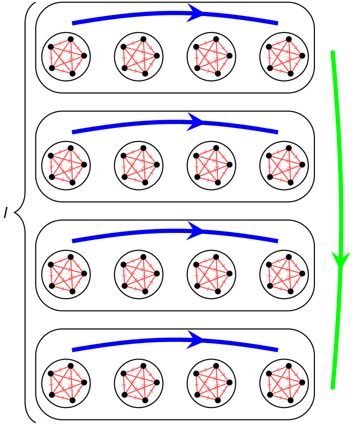

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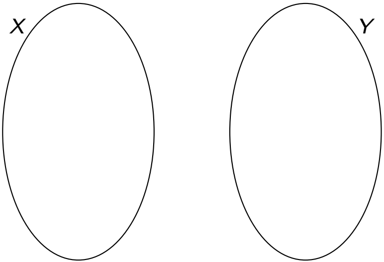

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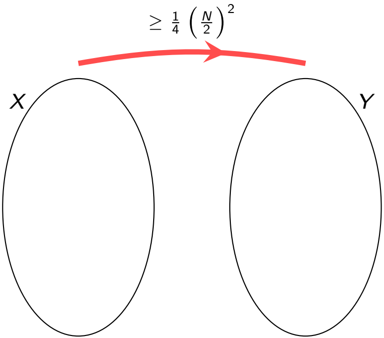

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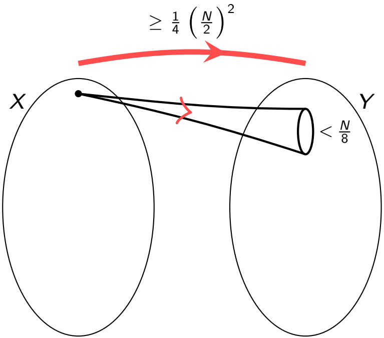

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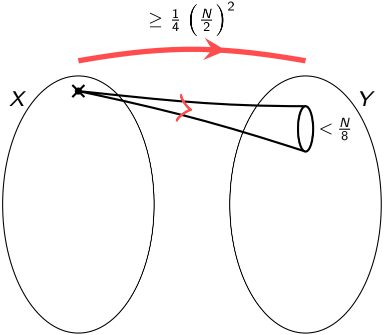

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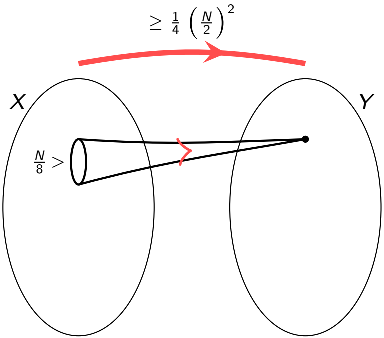

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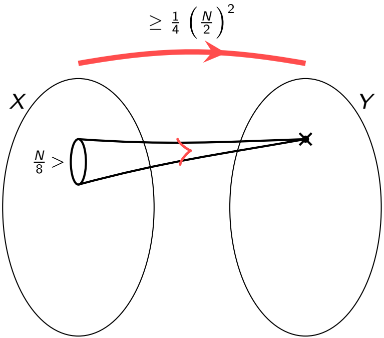

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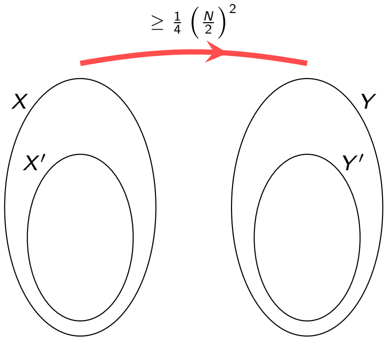

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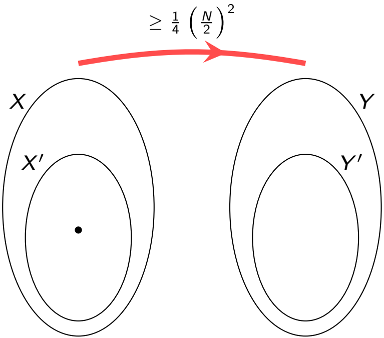

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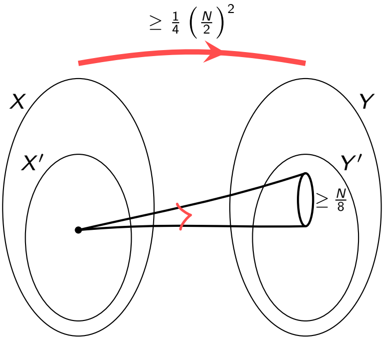

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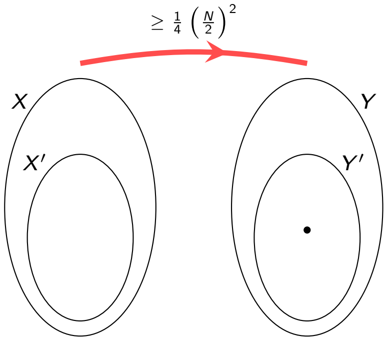

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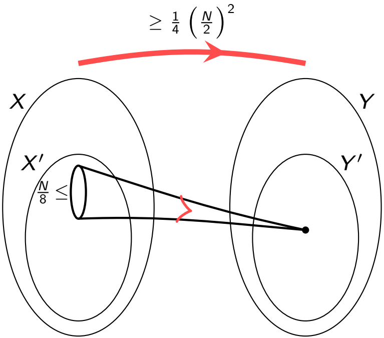

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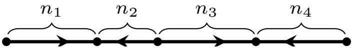
Recall, $N = 8nl + n$. $P =$ 

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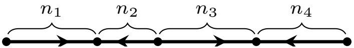
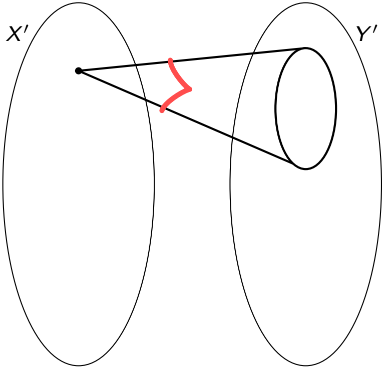
Recall, $N = 8nl + n$. $P =$ .



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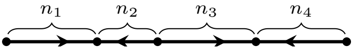
Recall, $N = 8nl + n$. $P =$ 

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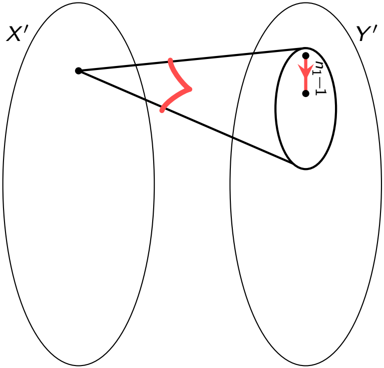

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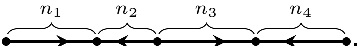
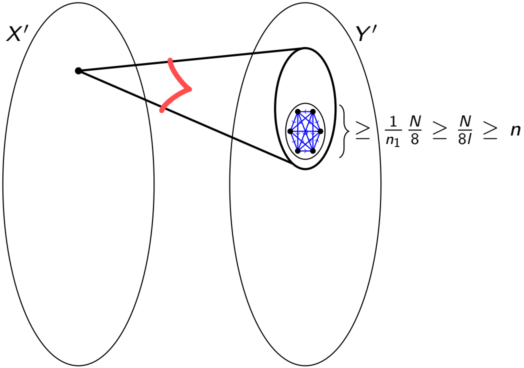
Recall, $N = 8nl + n$. $P =$ .



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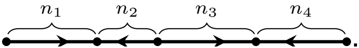
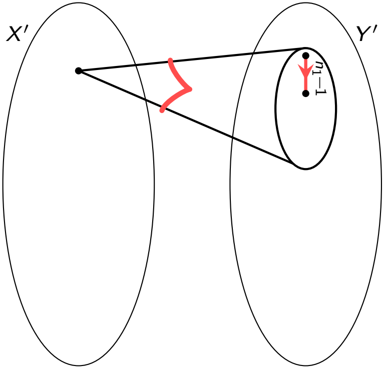
Recall, $N = 8nl + n$. $P =$ .



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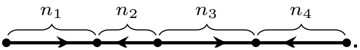
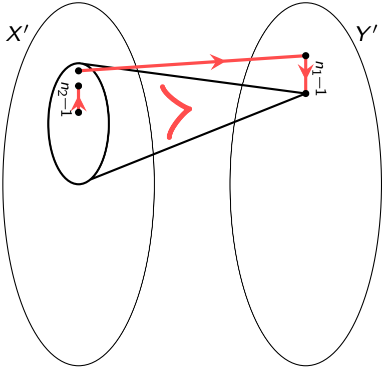
Recall, $N = 8nl + n$. $P =$ .



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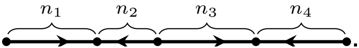
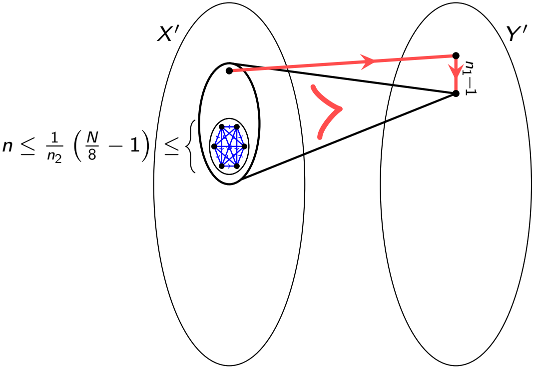
Recall, $N = 8nl + n$. $P =$ .



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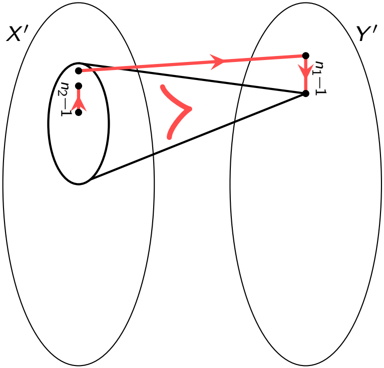

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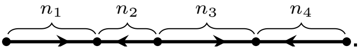
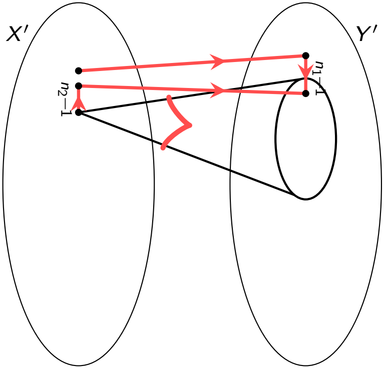
Recall, $N = 8nl + n$. $P =$ .



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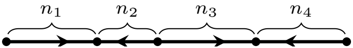
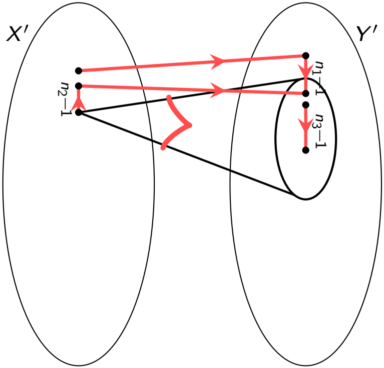
Recall, $N = 8nl + n$. $P =$ .



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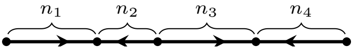
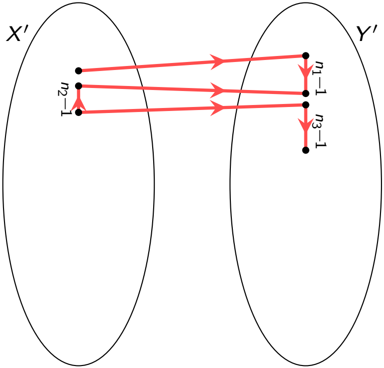
Recall, $N = 8nl + n$. $P =$ .



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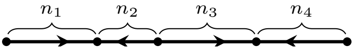
Recall, $N = 8nl + n$. $P =$ 

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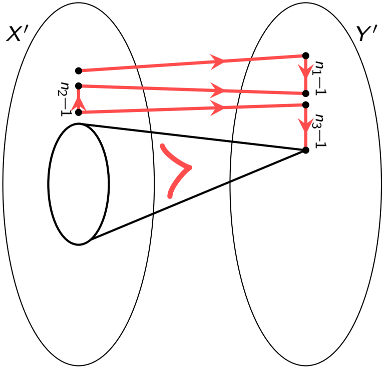

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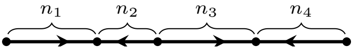
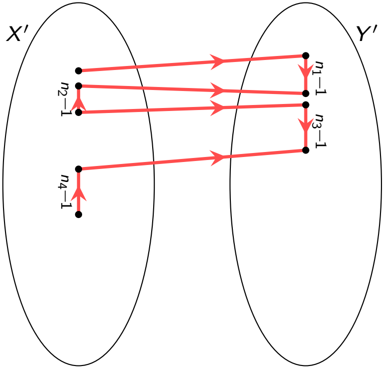
Recall, $N = 8nl + n$. $P =$ .



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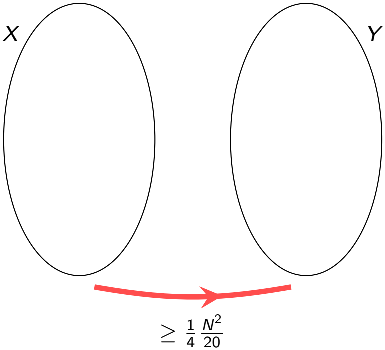

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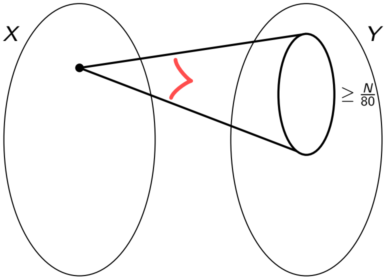

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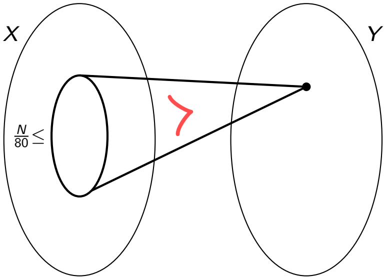

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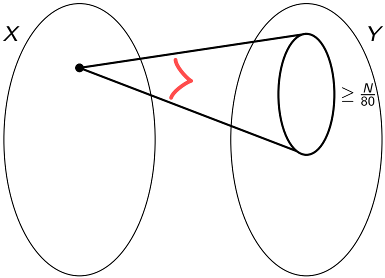

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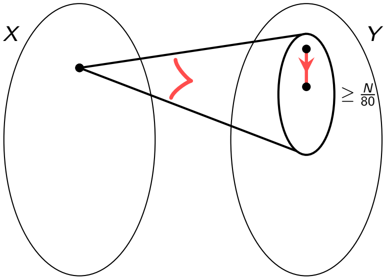

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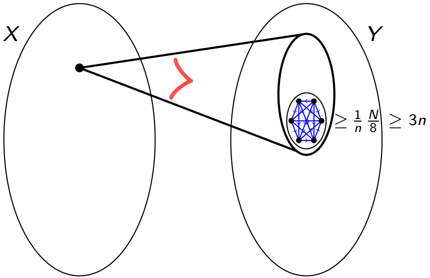

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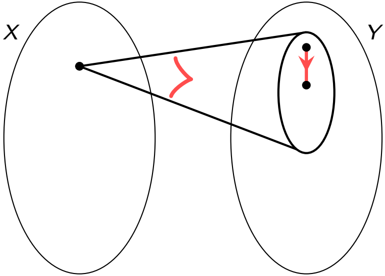

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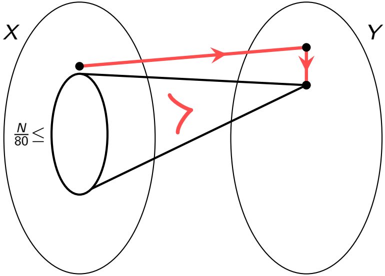

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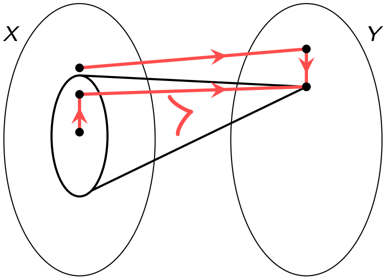

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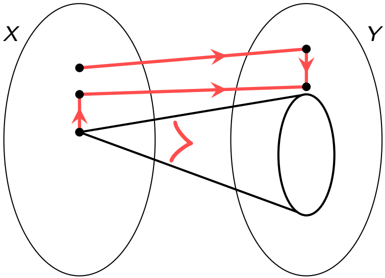

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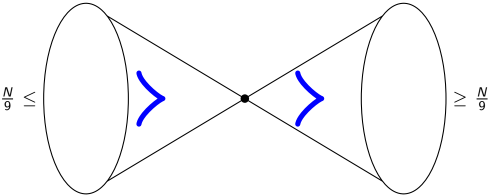

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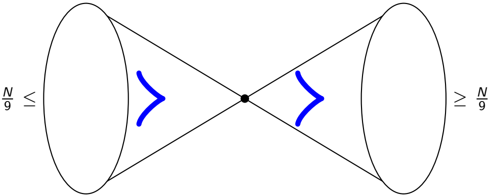

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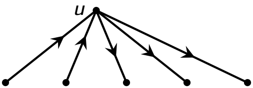

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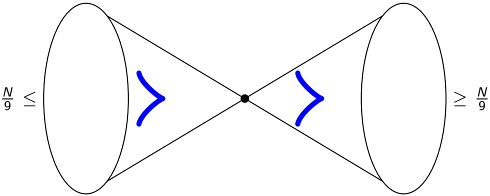

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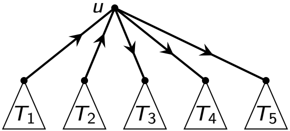

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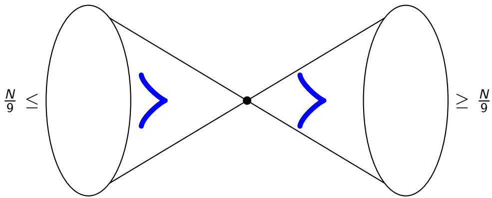

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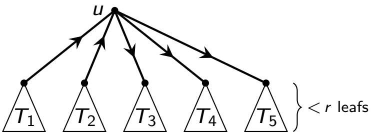

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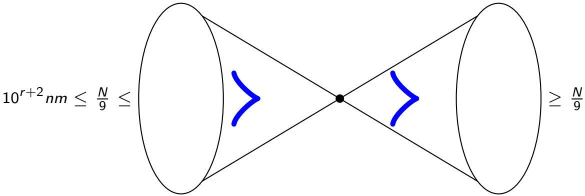

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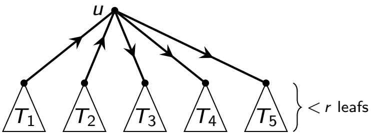

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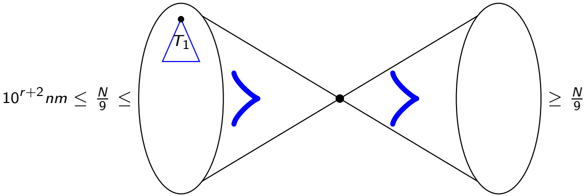

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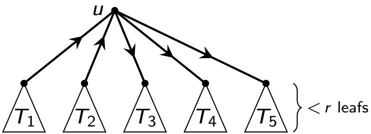

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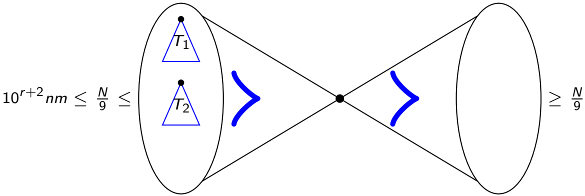

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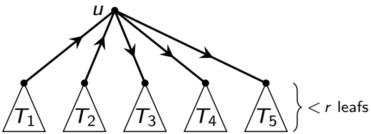

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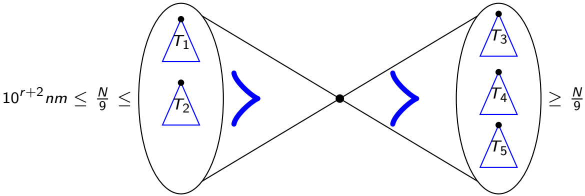

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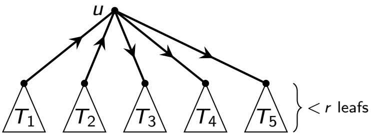

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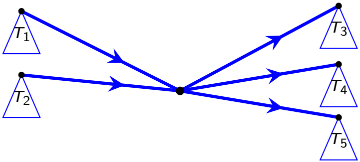

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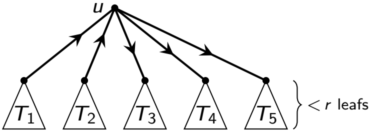

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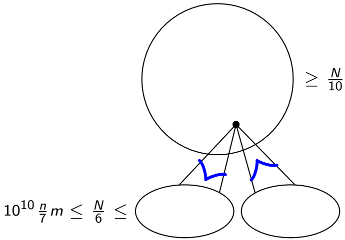

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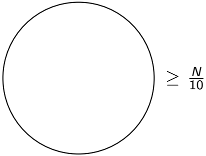

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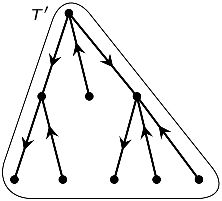

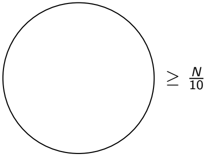

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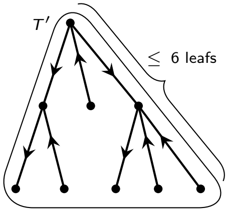

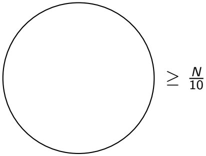

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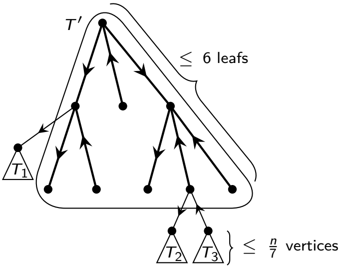

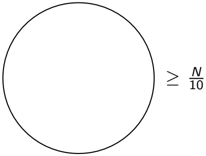

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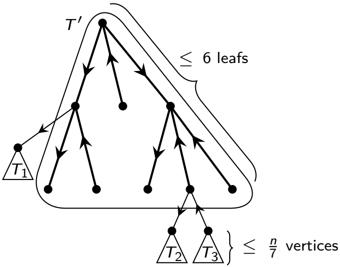

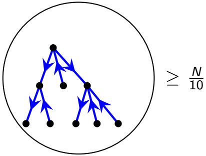

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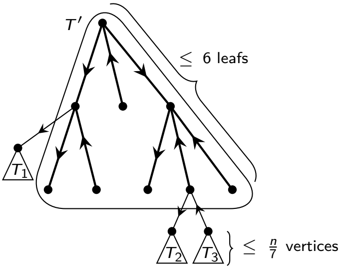

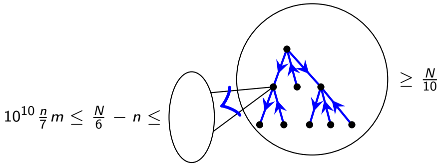

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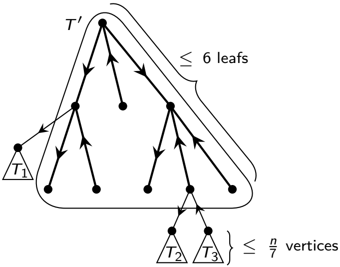

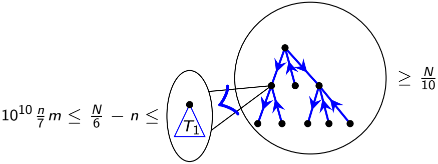

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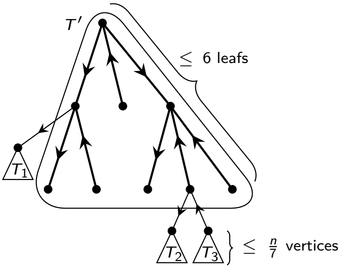

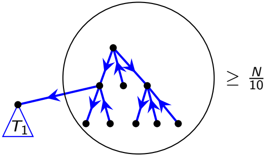

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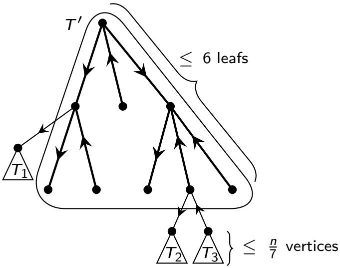

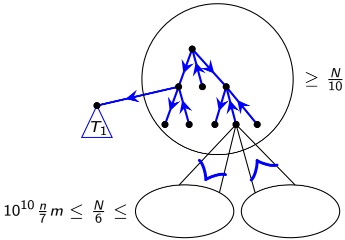

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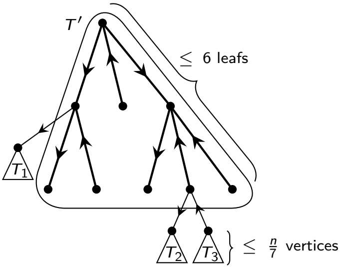

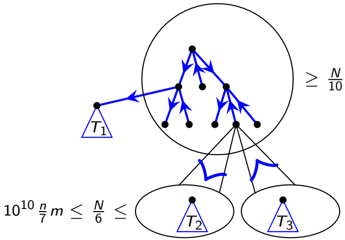

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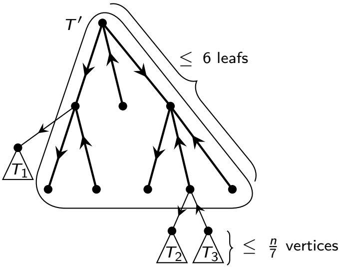

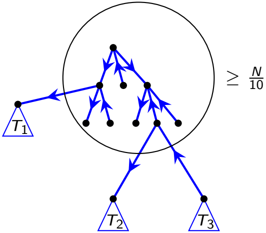

Tree vs. path - step 2

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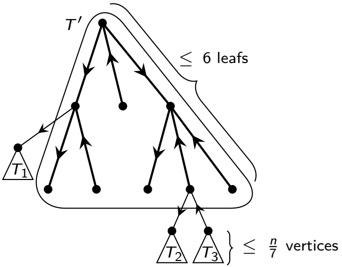

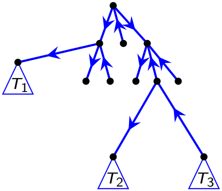

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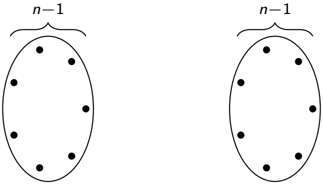

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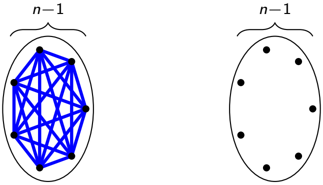

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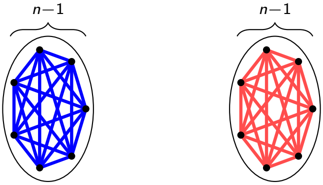

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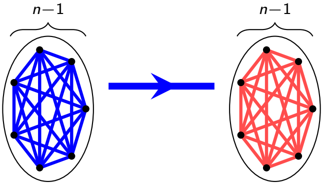

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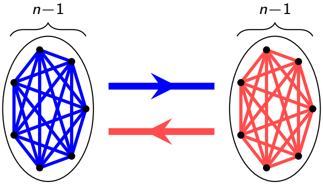

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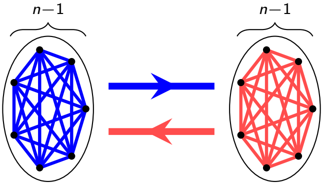

Using, GHRV, we conclude that $\overleftrightarrow{r}(\overrightarrow{P_{n+1}}, k) \le 2n^{k-1}$.

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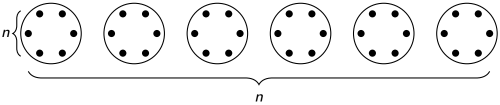

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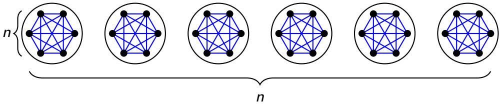

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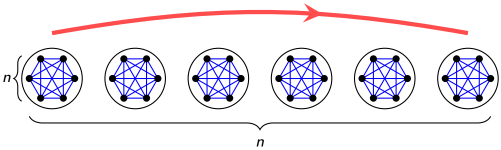

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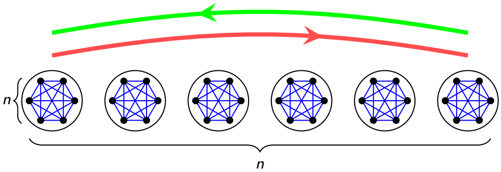

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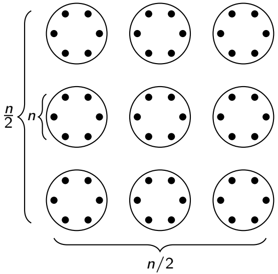

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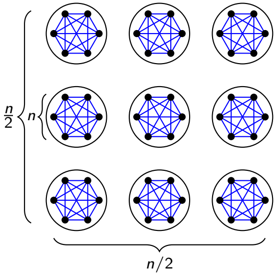

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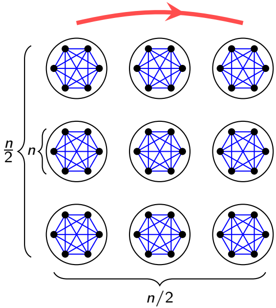

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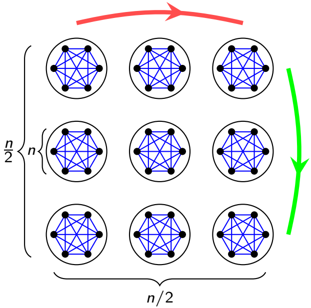

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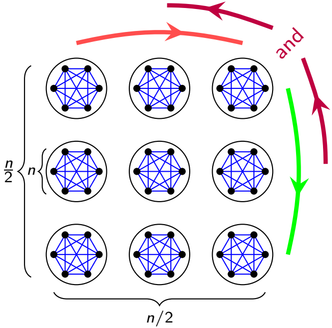

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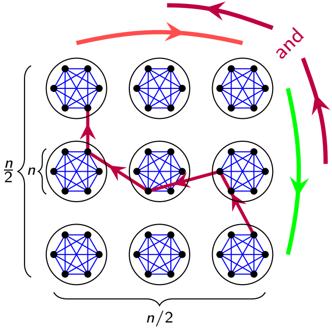

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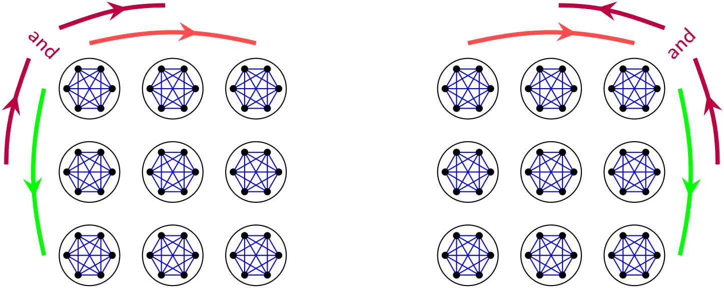

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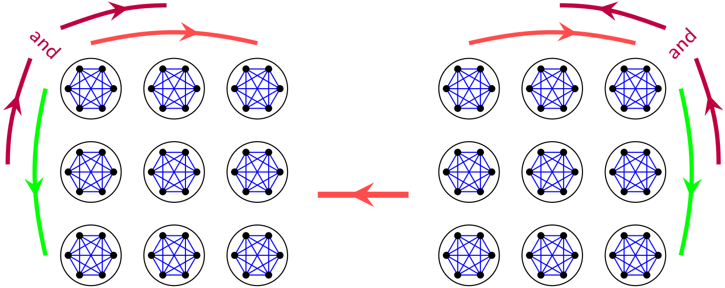

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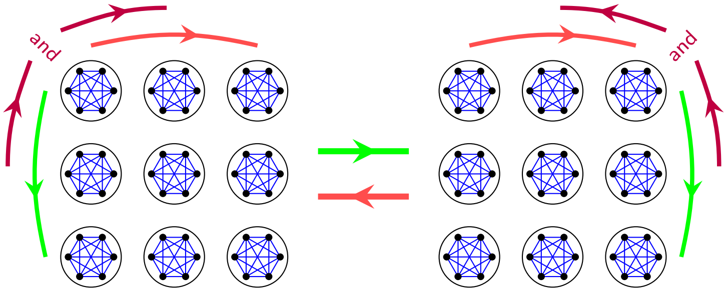

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The end

Thank you for listening!