Dense induced bipartite subgraphs in triangle-free graphs

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ETH - Institute for Theoretical Studies

joint work with Matthew Kwan, Benny Sudakov and Tuan Tran

Random Structures & Algorithms

July 2019

Max-Cut

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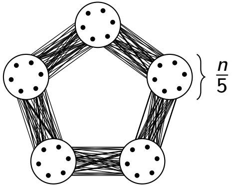

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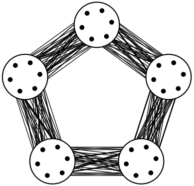

- **Erdős, Faudree, Pach, Spencer '88.** It suffices to remove $n^2/18$ edges.

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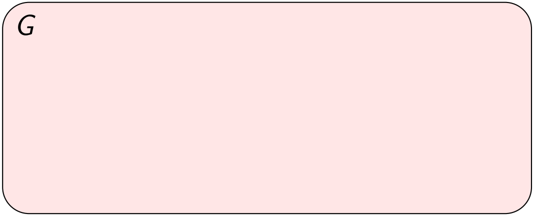

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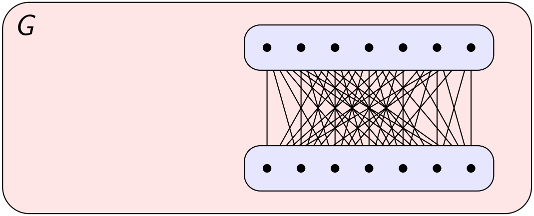

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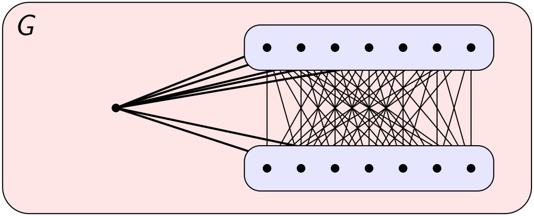

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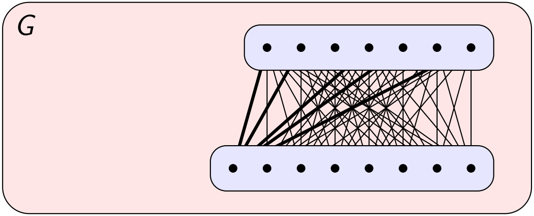

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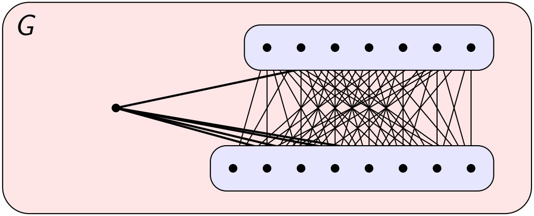

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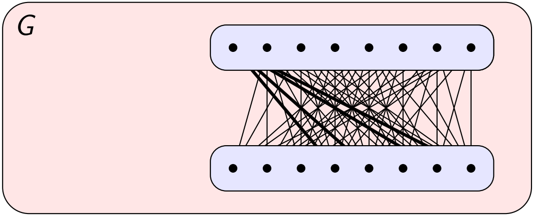

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- This would be tight: consider suitable random graphs.

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Theorem (Kwan, L., Sudakov, Tran '19+)

Every K_t-free graph with min degree d contains an induced bipartite subgraph with min degree at least $c_t \cdot \frac{\log d}{\log \log d}$.

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  Aim. Find disjoint independent sets A and B such that $G[A, B]$ has average degree $\Omega(\ell)$.

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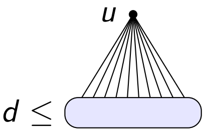

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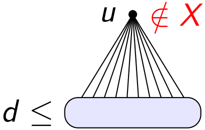

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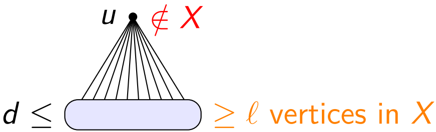

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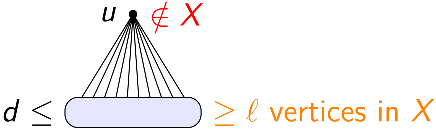

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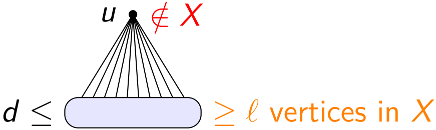

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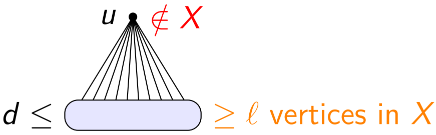

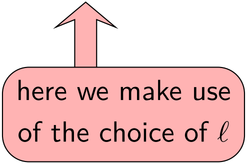

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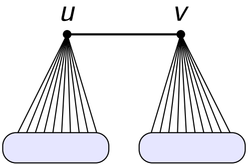

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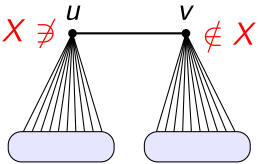

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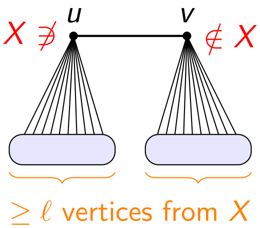

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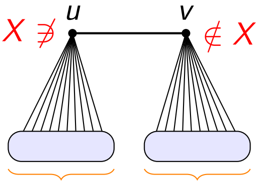

$\Rightarrow \mathbb{E}[e(Y)] \leq e(G) \cdot p^2 \leq \frac{n}{d}$.

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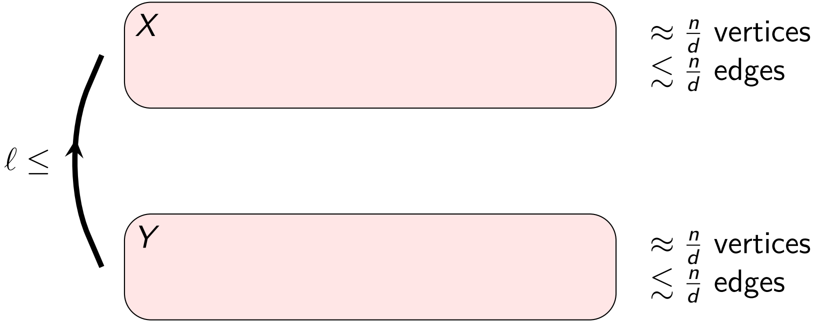

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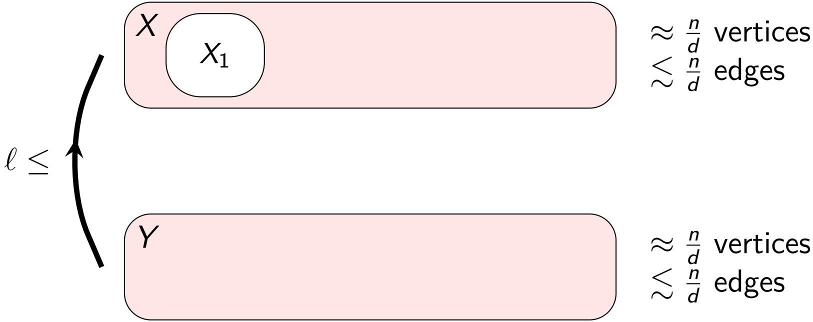

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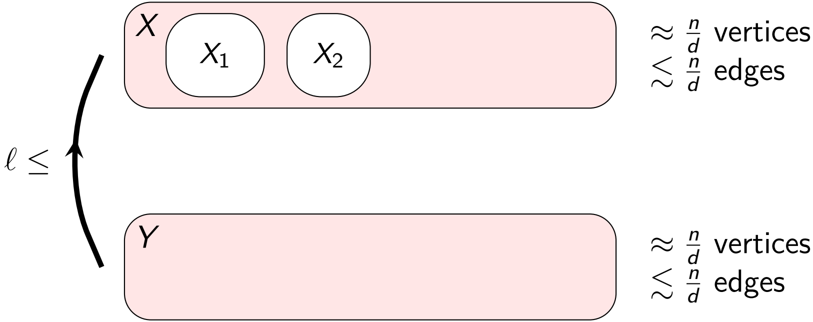

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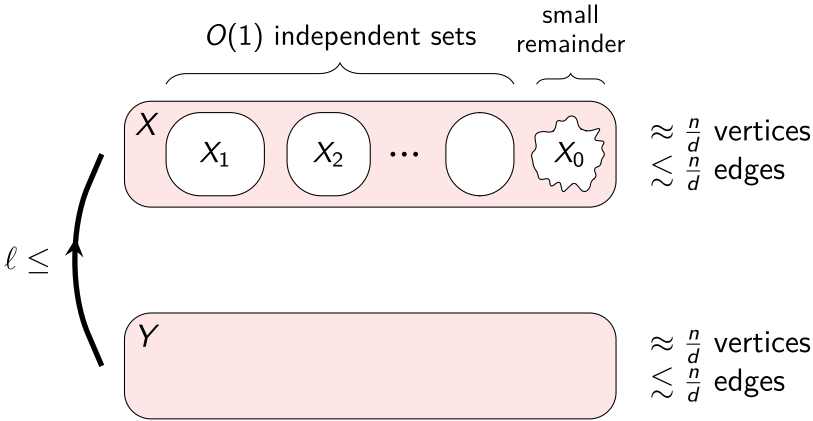

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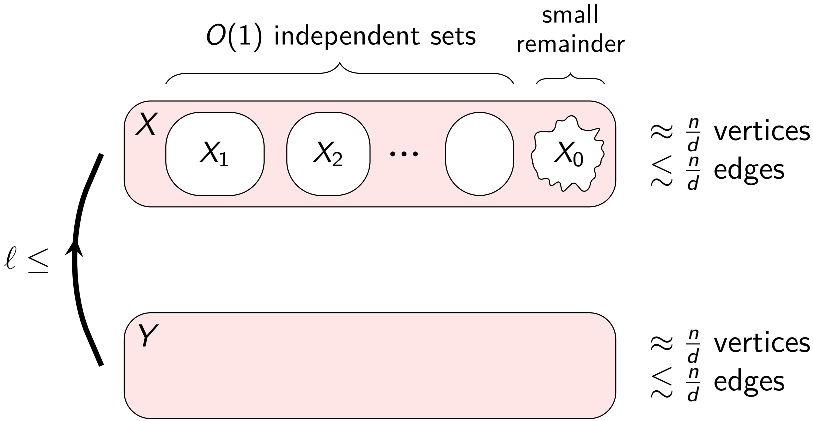

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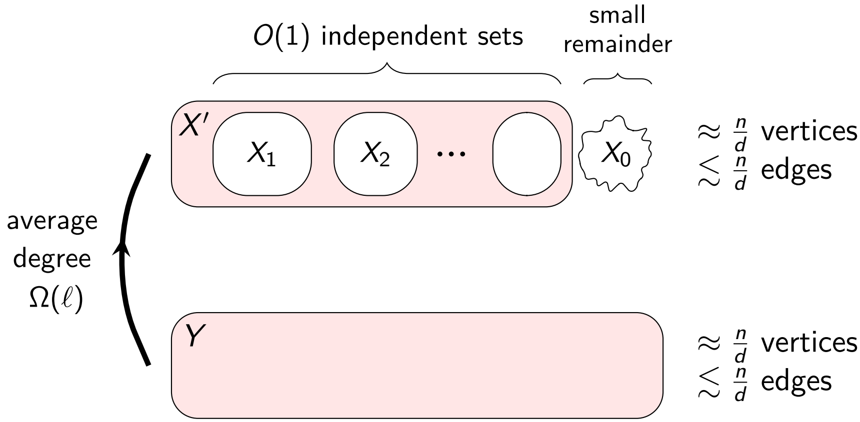

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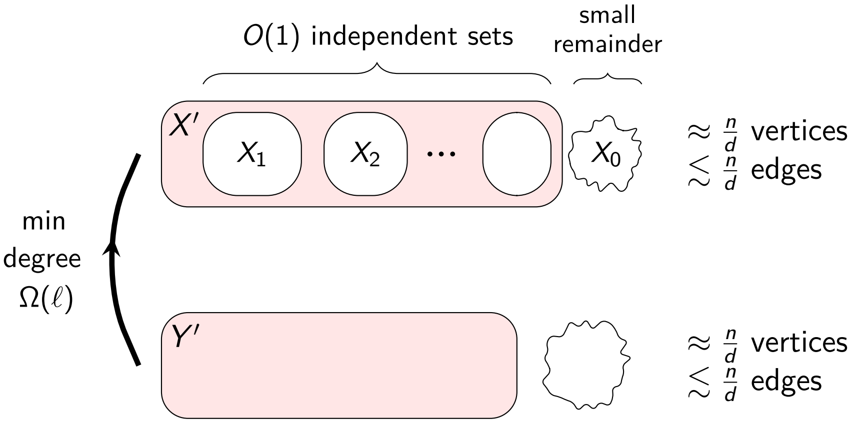

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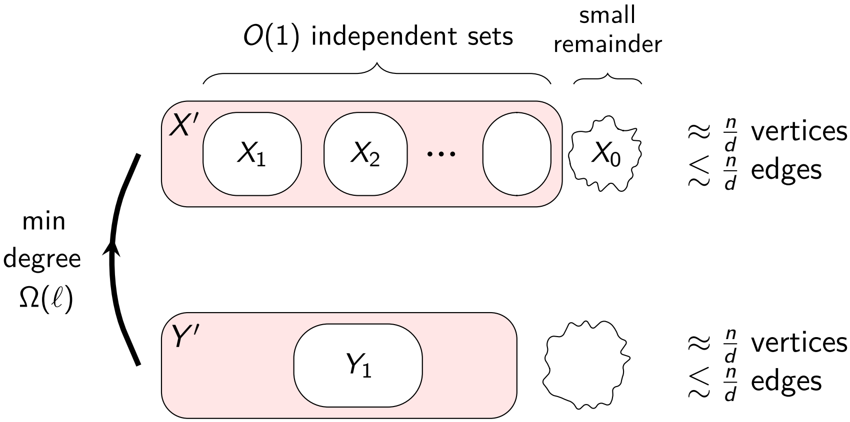

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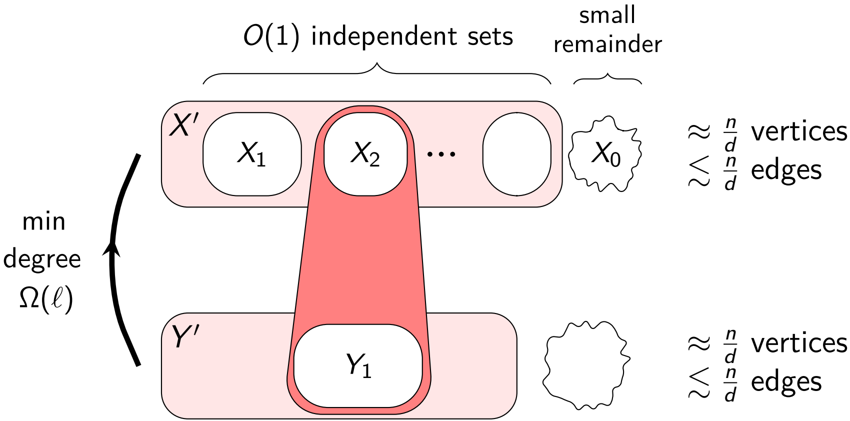

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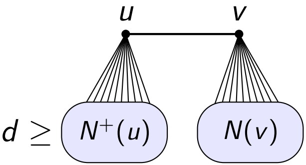

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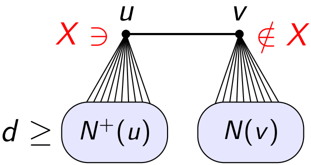

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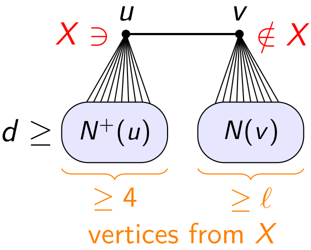

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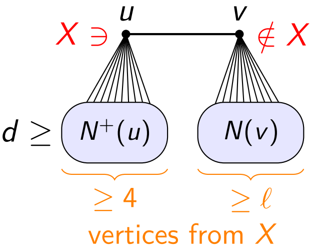
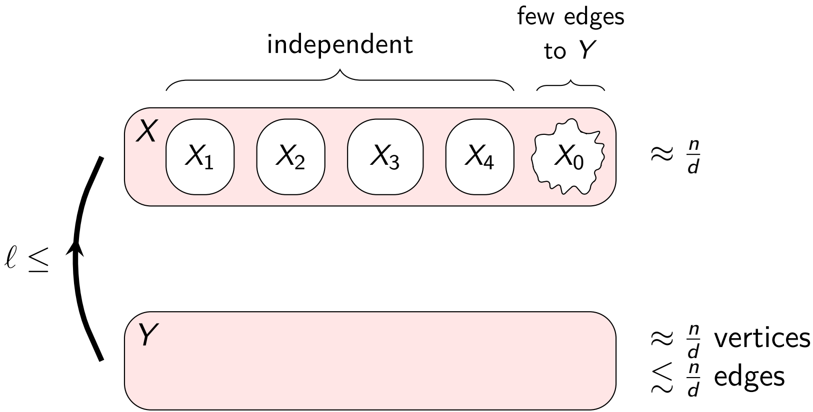


$\Rightarrow \mathbb{E}[e(X_0 - Y)] \leq e(G) \cdot \frac{p^2\ell}{23} \leq \frac{\ell}{23} \cdot \mathbb{E}[|Y|]$.

Completing the proof



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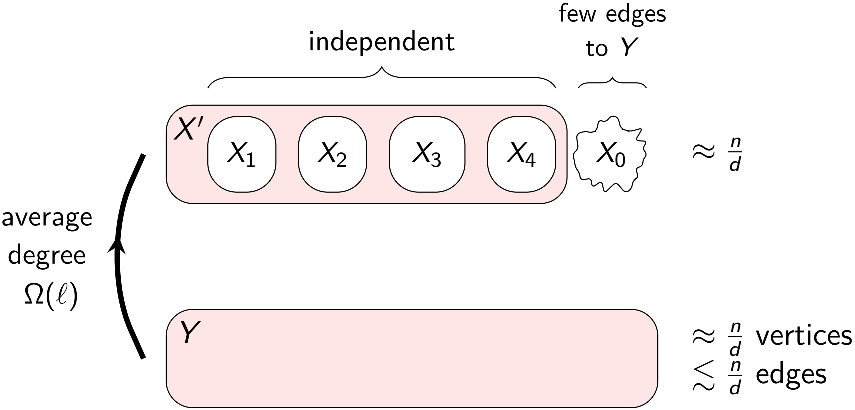

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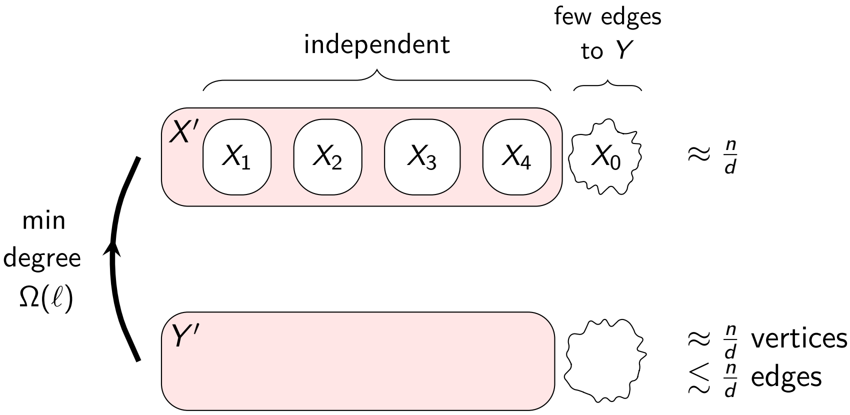

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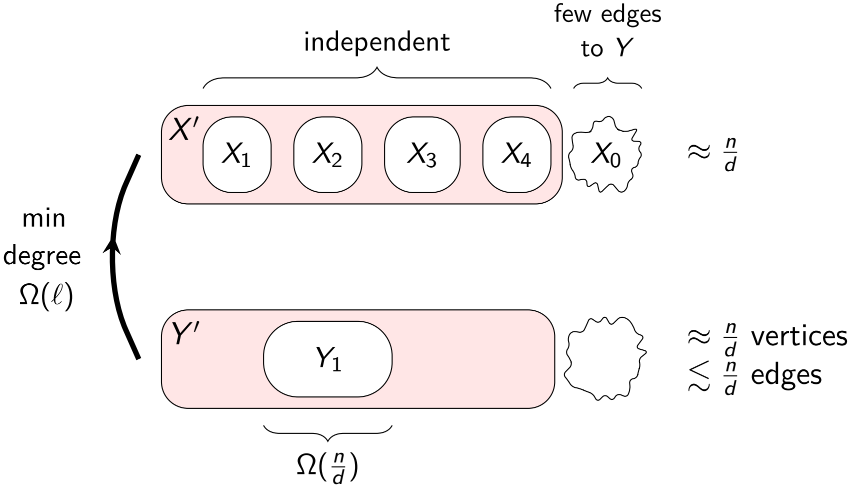

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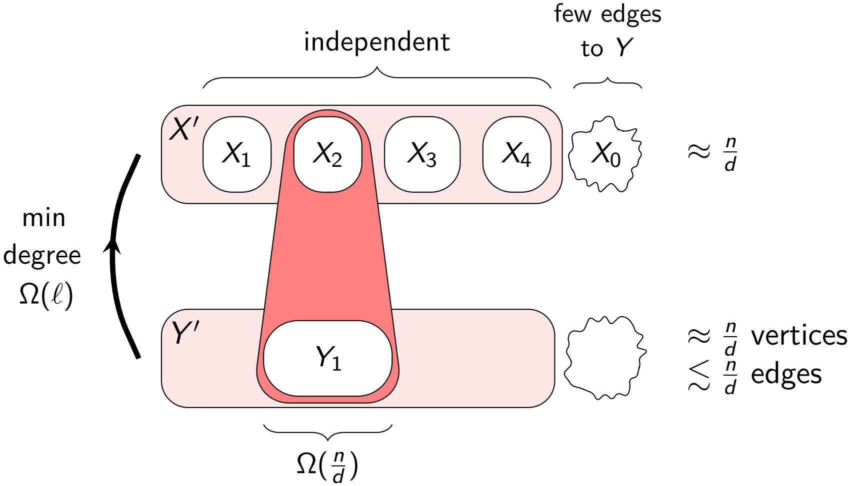

$\Rightarrow G[X_i, Y_1]$ has average degree $\Omega(\ell)$ for some i.

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- **Alon '94.** $g_{\triangle}(n, d) = \Theta(\frac{d^2}{n})$ if $d \geq n^{2/3}$.

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