Hypergraph Lagrangians

Shoham Letzter

ETH–ITS

joint work with Vytautas Gruslys and Natasha Morrison

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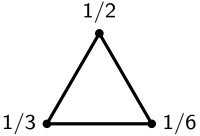

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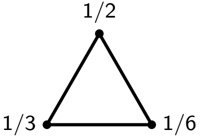

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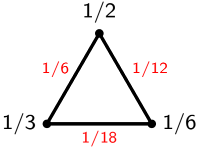

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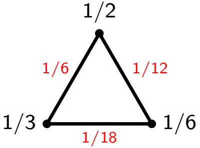

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The Lagrangian of G is

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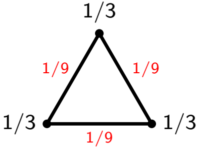

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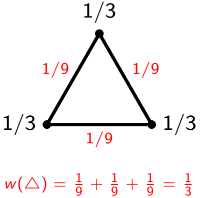

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More generally, $\lambda([t]^{(r)}) = \frac{1}{t^r}\binom{t}{r}$.

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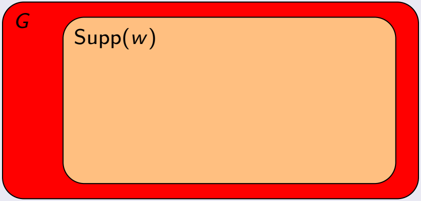

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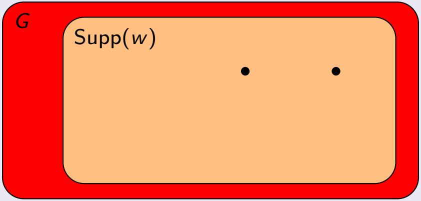

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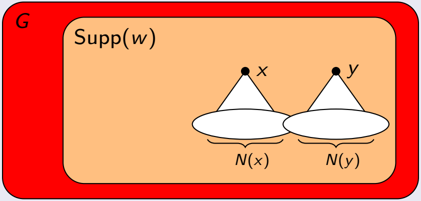

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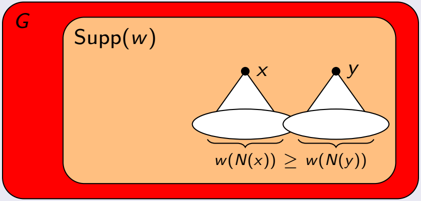

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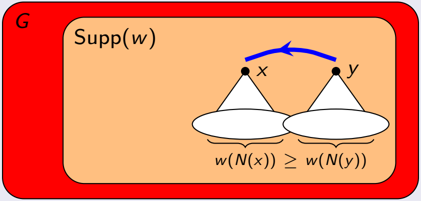

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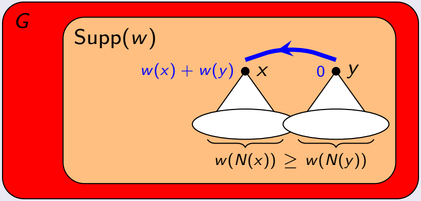

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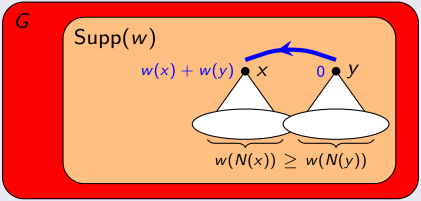

gain from weight shift:

$w(y)\Big(w(N(x)) - w(N(y))\Big) \geq 0$

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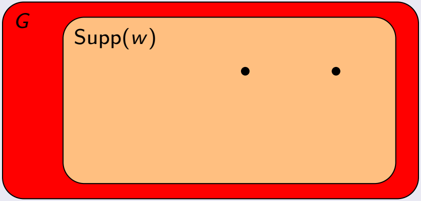

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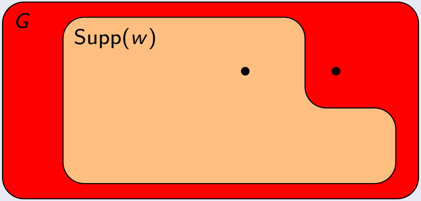

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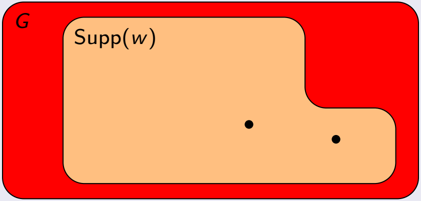

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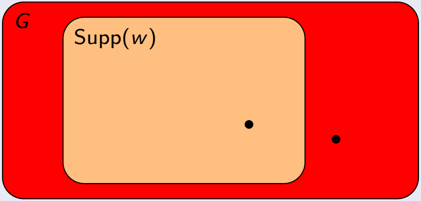

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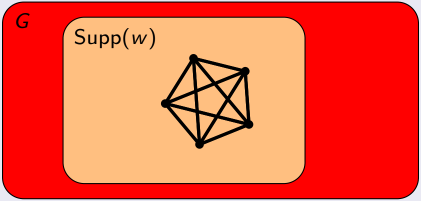

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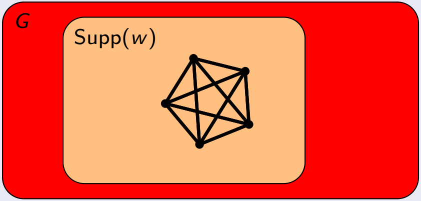

- Hence $\lambda(G) \leq \lambda(K_r)$.

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- **Frankl, Füredi '89; Hefetz, Keevash '13; ...**

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$\mathcal{C}(m, r)$ *is the initial segment of* $(\mathbb{N}^{(r)}, <_{\text{colex}})$ *of length* m*, where* $A <_{\text{colex}} B$ *iff* $\max(A \triangle B) \in B$.

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Conjecture (Frankl, Füredi '89)

$\mathcal{C}(m, r)$ *maximises* λ *among* r*-graphs with* m *edges.*

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- Improvements by **Tang, Peng, Zhang, Zhao ('15); Lei, Lu, Peng ('18)**.

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FF does not hold for all $r \geq 4$.

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Theorem (Gruslys, L., Morrison '19+)

The conjecture holds for $r \geq 4$ and $\binom{t}{r} \leq m \leq \binom{t}{r} + \binom{t-1}{r-1}$.

This extends the range of m's for which the conjecture is known to hold.

Theorem (Gruslys, L., Morrison '19+)

FF does not hold for all $r \geq 4$.
For example, it does not hold for $m = \binom{t}{r} + \binom{t-1}{r-1} + r$.

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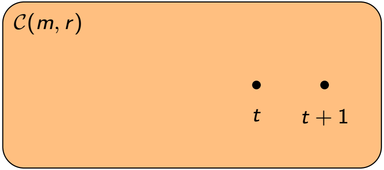

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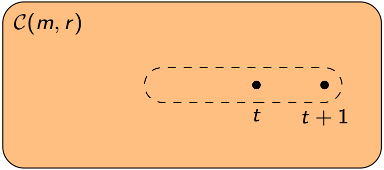

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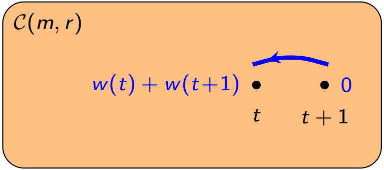

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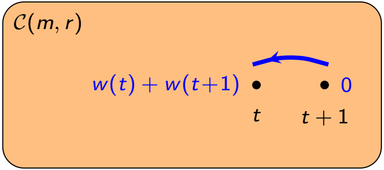

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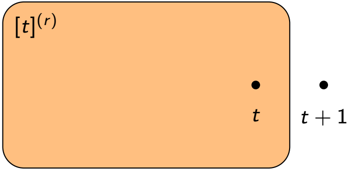

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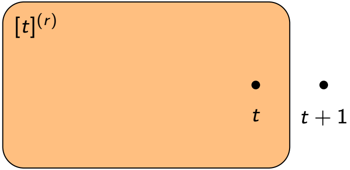

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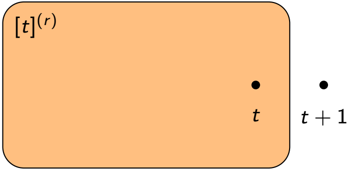

It follows that $\lambda(\mathcal{C}(m,r)) \leq \lambda([t]^{(r)})$ if $m \leq \binom{t}{r} + \binom{t-1}{r-1}$
(but for larger m we have $\lambda(\mathcal{C}(m,r)) > \lambda([t]^{(r)})$).

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Let $r = 4$, $m = \binom{t}{4} + \binom{t-1}{3} + 4$ and let $G := \mathcal{C}(m, 4)$.

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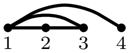

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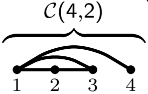

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We compare $\lambda(G)$ with $\lambda(G')$, where G' is defined as

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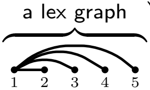

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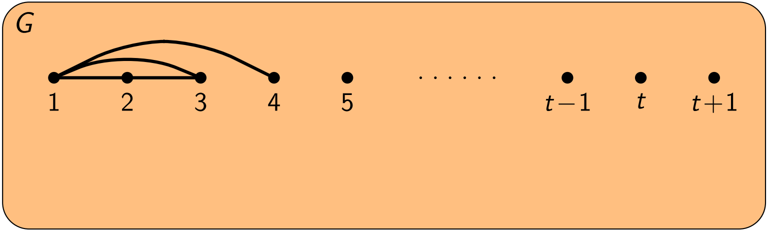

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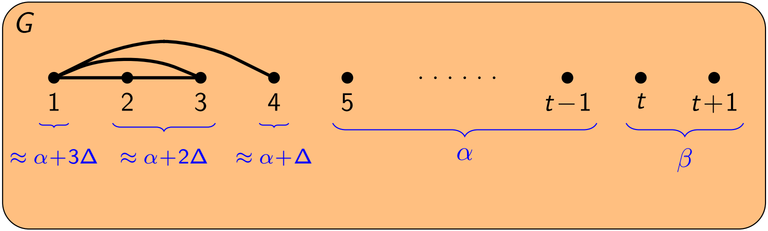

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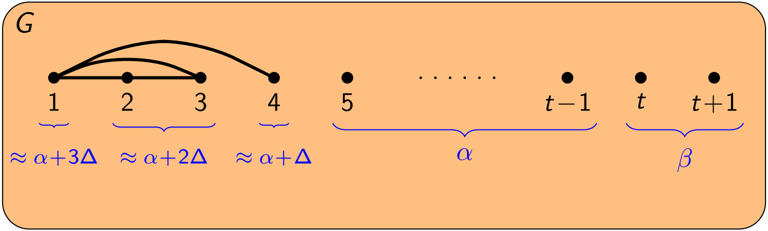

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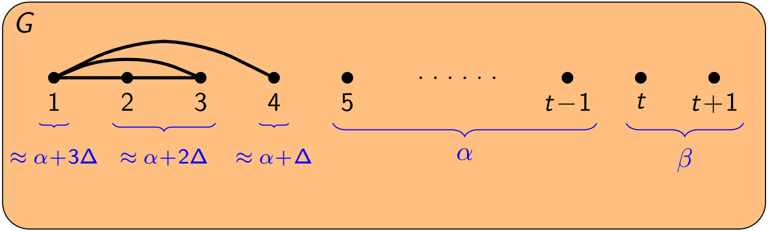

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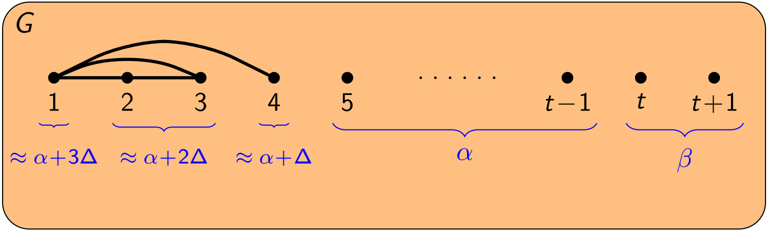

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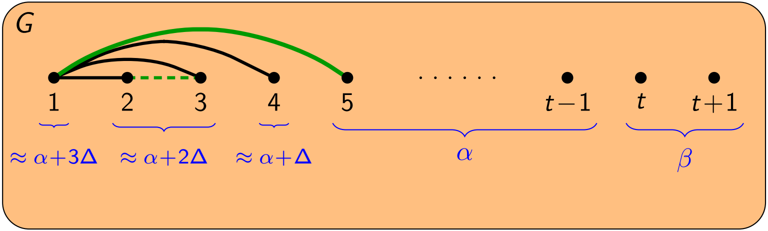

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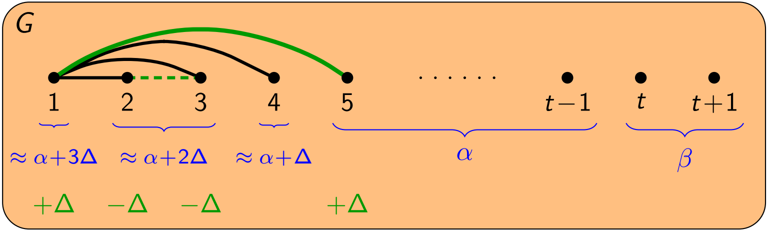

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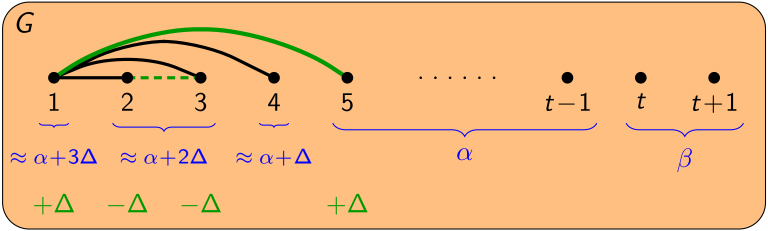

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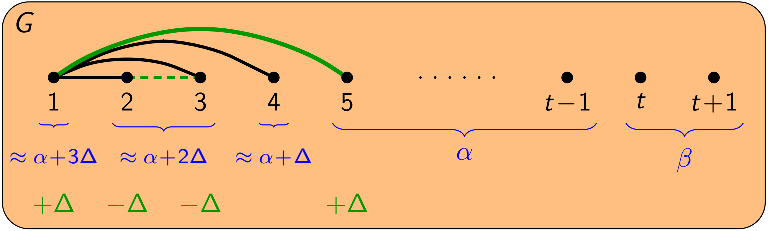

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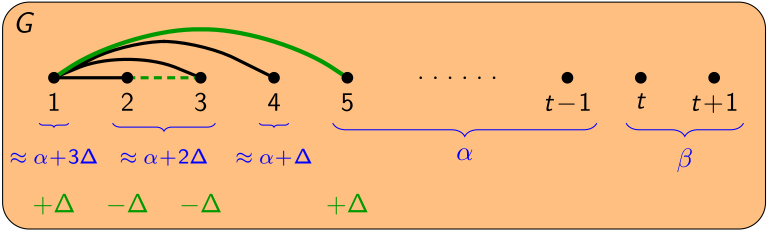

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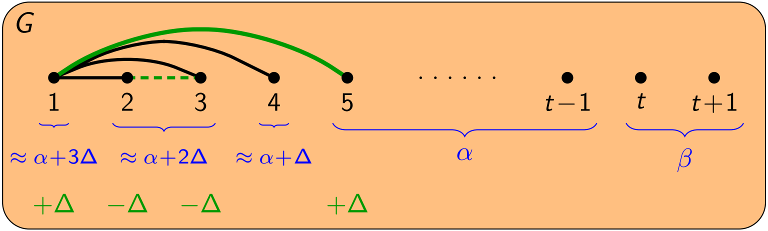

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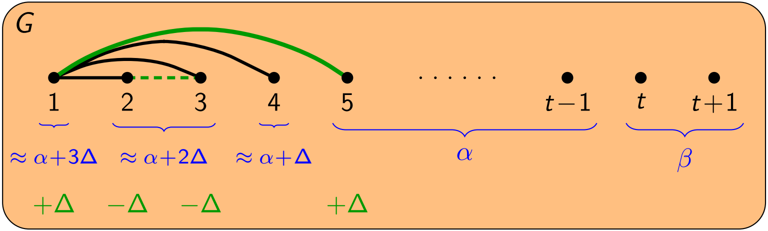

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- So, if $\mathcal{C}(s, r-2)$ is far from maximising sum of degrees squared, then m is a counterexample.

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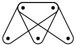
- For $3 \leq s \leq r-1$ there are two maximisers: $\mathcal{C}(s, r-2)$ and the star .

Remarks

Let $\binom{t}{r} + \binom{t-1}{r-1} \le m \le \binom{t+1}{r}$, $\quad m = \binom{t}{r} + \binom{t-1}{r-1} + s$.

We saw: if $\mathcal{C}(s, r-2)$ is far from maximising the sum of degrees squared, then m is a counterexample.

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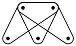
- For $3 \le s \le r-1$ there are two maximisers: $\mathcal{C}(s, r-2)$ and the star . We do not know which yields larger λ.

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Thank you for your attention!!!