Monochromatic triangle packings in red-blue graphs

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Probabilistic and Extremal Combinatorics

DMV Jahrestagung

September 2020

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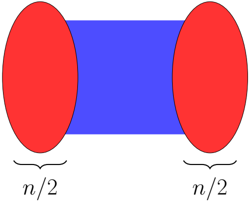
- For even n,  is extremal.

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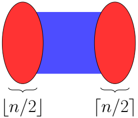
- The conjecture is tight: 

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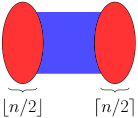
- The conjecture is tight: 

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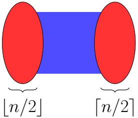
- The conjecture is tight:  .

- **Erdős–Faudree–Gould–Jacobson–Lehel '01.** Every red-blue K_n has $\frac{3n^2}{55} + o(n^2)$ edge-disjoint mono triangles.
- **Keevash–Sudakov '04.** Every red-blue K_n has $\frac{n^2}{12.89} + o(n^2)$ edge-disjoint mono triangles.

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  Moreover, a **stability** result holds: either $\overline{G}$ is εn^2-close to bipartite, or G has $\geq \frac{n^2}{12} + \delta n^2$ edge-disjoint triangles.

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For every $\varepsilon > 0$ there is $\delta > 0$ s.t. in every red-blue K_n,

- *either one of the colours is εn^2-close to bipartite,*
- *or there are $\geq \frac{n^2}{12} + \delta n^2$ edge-disjoint mono triangles.*

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A **fractional $\triangle$-packing** in G is a function $\omega : \{\text{triangles in } G\} \to [0,1]$ s.t. for every edge xy:

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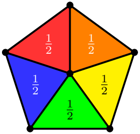

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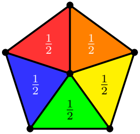

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$$\nu(G) = \max \left\{ 3 \sum_{xyz \text{ is a triangle}} \omega(xyz) : \begin{array}{l} \omega \text{ a fractional} \\ \triangle\text{-packing} \end{array} \right\}$$

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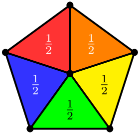

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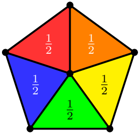

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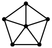
- $\nu($  $) = 3 \cdot 5 \cdot 1/2.$

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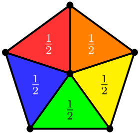

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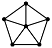
- $\nu($  $) = 3 \cdot 5 \cdot 1/2$.
- $\nu(K_n) = \binom{n}{2}$ for $n \neq 2$.

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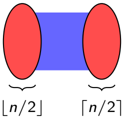
- $\nu_{\text{mono}}\Big($ 

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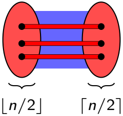
- $\nu_{\text{mono}}\Bigg($ 

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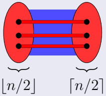
Theorem (Gruslys–L. '20)

Let G be a red-blue K_n, with $n \geq 22$. Then $\nu_{\text{mono}}(G) \geq \lfloor \frac{(n-1)^2}{4} \rfloor$, with equality iff $G =$ 

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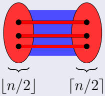
Let G be a red-blue K_n, with $n \geq 22$. Then $\nu_{\text{mono}}(G) \geq \lfloor \frac{(n-1)^2}{4} \rfloor$, with equality iff $G =$ 

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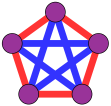
- A **pentagon blow-up** is .

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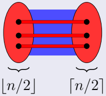
Let G be a red-blue K_n, with $n \geq 22$. Then $\nu_{\text{mono}}(G) \geq \lfloor \frac{(n-1)^2}{4} \rfloor$, with equality iff $G =$ 

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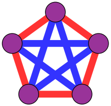
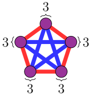
- A **pentagon blow-up** is  .

$\nu_{\text{mono}}\Big($ 

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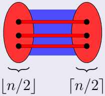
Let G be a red-blue K_n, with $n \geq 22$. Then $\nu_{\text{mono}}(G) \geq \lfloor \frac{(n-1)^2}{4} \rfloor$, with equality iff $G =$ 

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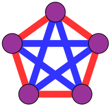
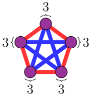
- A **pentagon blow-up** is  .

  $\nu_{\text{mono}}\Big($ 

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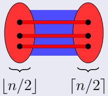
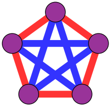
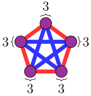
Let G be a red-blue K_n, with $n \geq 22$. Then $\nu_{\text{mono}}(G) \geq \lfloor \frac{(n-1)^2}{4} \rfloor$, with equality iff $G =$  *.*

- A **pentagon blow-up** is  .

  $\nu_{\text{mono}}\Big($  $\Big) = 45 < 49 = \lfloor \frac{14^2}{4} \rfloor$.

- **Haxell–Rödl '01.** Packing number ≈ fractional packing number. Hence: every red-blue K_n has $\approx \frac{n^2}{12}$ edge-disjoint mono triangles.

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Let G be a red-blue K_n, where $n \geq 26$. If $\nu_{\text{mono}}(G) \leq \frac{n(n-1)}{4}$, then one of the colours is $n/8$-close to bipartite.

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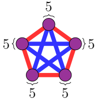
- $\nu_{\text{mono}}\Bigg($ 

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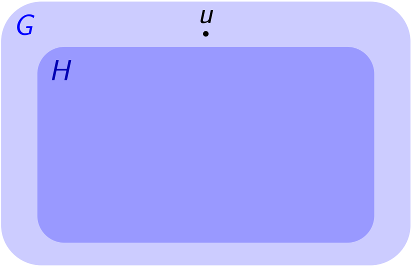

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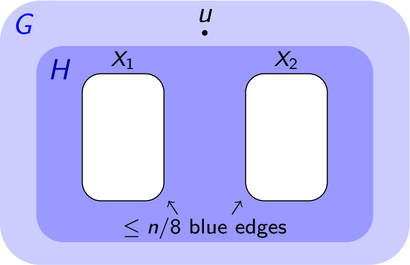

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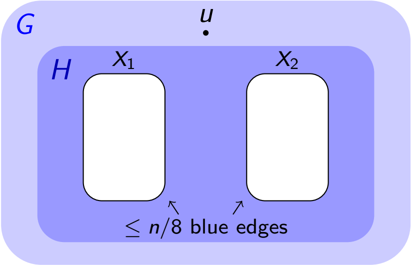
- **Aim:** the blue graph in G is $(n+1)/8$-close to bipartite.



Induction step

- Suppose G red-blue K_{n+1} and $\nu_{\text{mono}}(G) \le \frac{n(n+1)}{4}$.
- By proposition, $\nu_{\text{mono}}(G \setminus u) \le \frac{n(n-1)}{4}$ for some vertex u.
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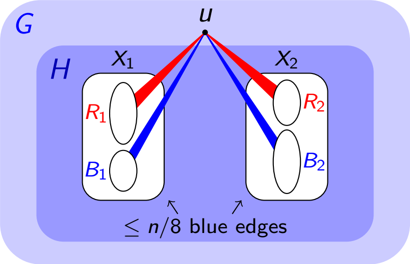
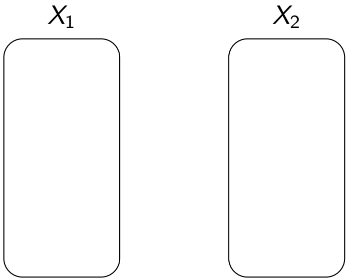


Covering blue edges in X_1 and X_2

Aim. cover blue edges in X_1,X_2 by disjoint blue cross triangles.



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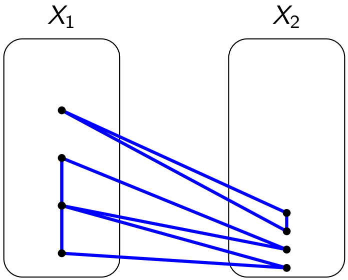

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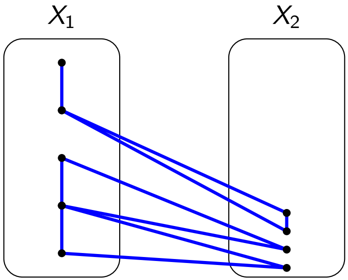

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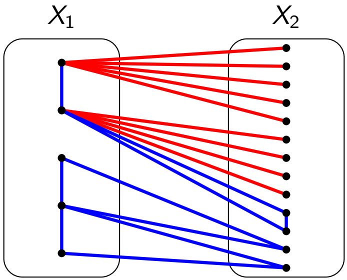

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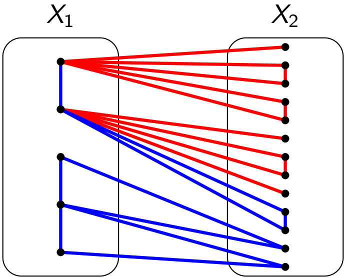

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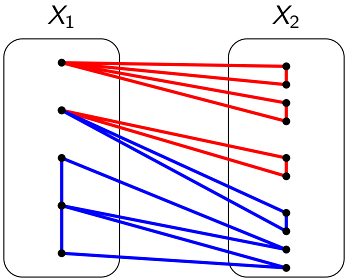

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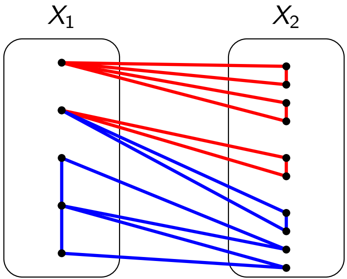

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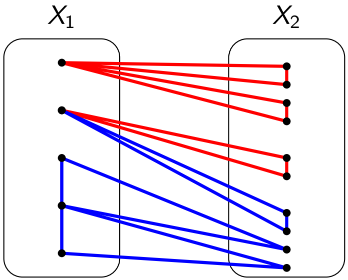

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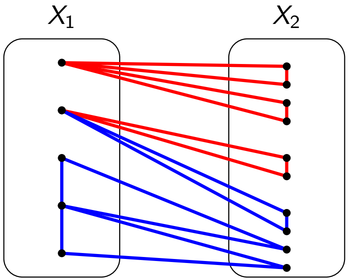

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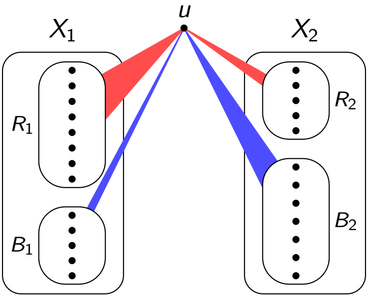

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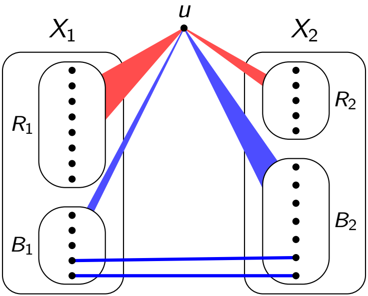

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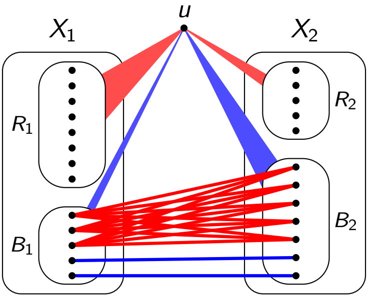

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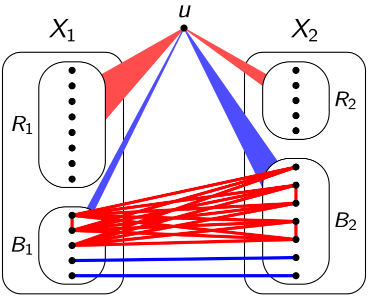

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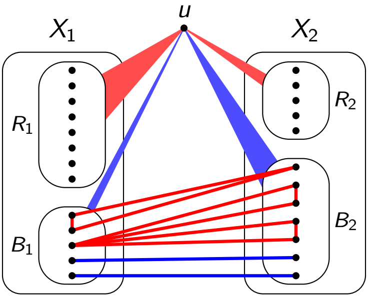

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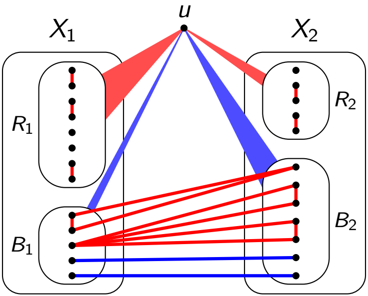

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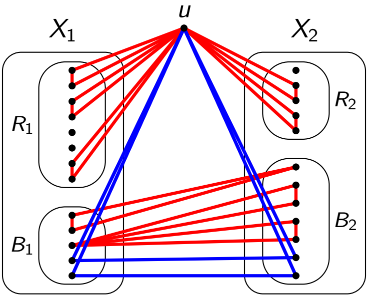

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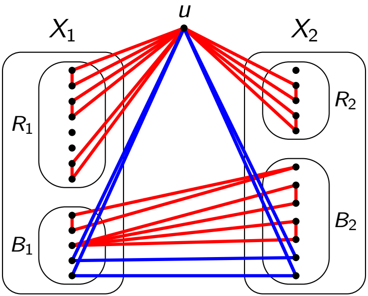

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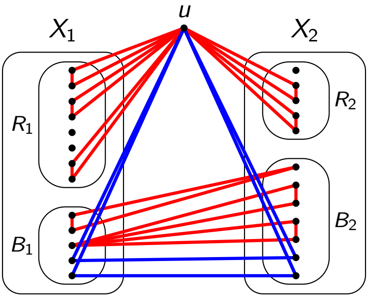

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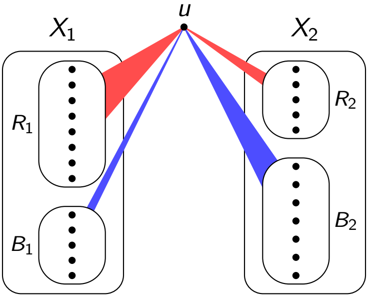

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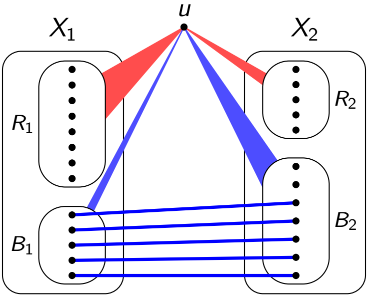

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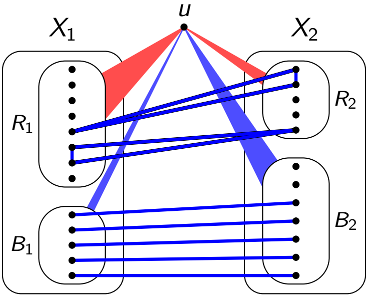

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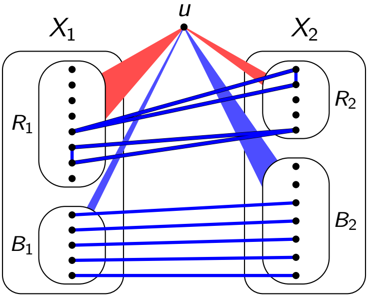

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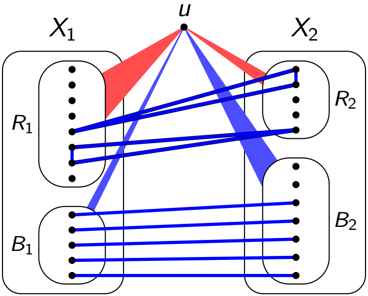

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Thank you for listening!!!