Directed cycles with zero weight in $\mathbb{Z}_p^k$

Shoham Letzter
UCL

Joint work with Natasha Morrison

Ascona

August 2022

Zero-sum Ramsey

Erdős-Ginzburg-Ziv '61: every sequence of $2n-1$ elements in $\mathbb{Z}_n$ contains a subsequence of length n whose elements sum to 0.

Zero-sum Ramsey

Erdős-Ginzburg-Ziv '61: every sequence of $2n-1$ elements in $\mathbb{Z}_n$ contains a subsequence of length n whose elements sum to 0.

Further directions:

* For abelian A, what is the min ℓ s.t. every sequence of ℓ elements in A contain a non-empty subsequence summing to 0?

Zero-sum Ramsey

Erdős-Ginzburg-Ziv '61: every sequence of $2n-1$ elements in $\mathbb{Z}_n$ contains a subsequence of length n whose elements sum to 0.

Further directions:

* For abelian A, what is the min ℓ s.t. every sequence of ℓ elements in A contain a non-empty subsequence summing to 0?
  - This is the Davenport constant of A.

Zero-sum Ramsey

Erdős-Ginzburg-Ziv '61: every sequence of $2n-1$ elements in $\mathbb{Z}_n$ contains a subsequence of length n whose elements sum to 0.

Further directions:

* For abelian A, what is the min ℓ s.t. every sequence of ℓ elements in A contain a non-empty subsequence summing to 0?
    - This is the **Davenport constant** of A.
    - Solved by **Olson '69** when $|A| = p^k$ for prime p.

Zero-sum Ramsey

* The <u>zero-sum Ramsey number</u> $r_{zs}(H, \mathbb{Z}_k)$ is the min n s.t. for every $\mathbb{Z}_k$-edge-colouring of K_n there is a zero-sum copy of H.

Zero-sum Ramsey

* The zero-sum Ramsey number $r_{zs}(H, \mathbb{Z}_k)$ is the min n s.t. for every $\mathbb{Z}_k$-edge-colouring of K_n there is a zero-sum copy of H.

  - introduced by Bialostocki–Dierker '92.

Zero-sum Ramsey

* The zero-sum Ramsey number $r_{zs}(H, \mathbb{Z}_k)$ is the min n s.t. for every $\mathbb{Z}_k$-edge-colouring of K_n there is a zero-sum copy of H.

  - introduced by Bialostocki–Dierker '92.
  - finite when $k \mid e(H)$.

Zero-sum Ramsey

* The zero-sum Ramsey number $r_{zs}(H, \mathbb{Z}_k)$ is the min n s.t. for every $\mathbb{Z}_k$-edge-colouring of K_n there is a zero-sum copy of H.

- introduced by Bialostocki–Dierker '92.

- finite when $k \mid e(H)$.

- investigated for matchings, complete graphs, trees ...
  by Alon, Bialostocki, Caro, Chung, Dierker, Füredi, Graham, Kleitman, Roditty, Seymour, Schrijver ...

Zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \longrightarrow A \ \exists \text{ zero-sum cycle}\}$.

complete digraph on n vertices

Zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \to A \ \exists \text{ zero-sum cycle}\}$.

complete digraph on n vertices

* Introduced by **Alon-Krivelevich '21** for $A = \mathbb{Z}_q$.

Zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \longrightarrow A \ \exists \text{ zero-sum cycle}\}$.

complete digraph on n vertices

* Introduced by **Alon-Krivelevich '21** for $A = \mathbb{Z}_q$.

  They were interested in finding cycles of length $0 \pmod q$ in clique minors.

Zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \longrightarrow A \ \exists \text{ zero-sum cycle}\}$.

complete digraph on n vertices

* Introduced by **Alon-Krivelevich '21** for $A = \mathbb{Z}_q$.

  They were interested in finding cycles of length $0 \pmod q$ in clique minors.

* $f(\mathbb{Z}_q) \geq q$ (when $n = q-1$, consider $\omega \equiv 1$).

Previous results on zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \to A \ \exists \text{ zero-sum cycle}\}$.

* <u>Alon-Krivelevich '21</u>: $f(\mathbb{Z}_q) = O(q \log q)$,
  $f(\mathbb{Z}_p) \leq 2p$ for p prime.

Previous results on zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \to A \ \exists \text{ zero-sum cycle}\}$.

* **Alon-Krivelevich '21:** $f(\mathbb{Z}_q) = O(q \log q)$,
  $f(\mathbb{Z}_p) \leq 2p$ for p prime.

* **Mészarós-Steiner '22:** $f(A) \leq 8|A|$ for abelian A,
  $f(\mathbb{Z}_p) \leq \frac{3p}{2}$ for p prime.

Previous results on zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \to A \ \exists \text{ zero-sum cycle}\}$.

* **Alon-Krivelevich '21:** $f(\mathbb{Z}_q) = O(q \log q)$,
  $f(\mathbb{Z}_p) \leq 2p$ for p prime.

* **Mészarós-Steiner '22:** $f(A) \leq 8|A|$ for abelian A,
  $f(\mathbb{Z}_p) \leq \frac{3p}{2}$ for p prime.

* **Berendsohn-Boyadzhiyska-Kozma '22:** $f(A) \leq 2|A|$ for any group A.

Previous results on zero-sum cycles

For abelian A, $f(A) := \min\{n : \forall \omega : E(\vec{K}_n) \to A \ \exists \text{ zero-sum cycle}\}$.

* **Alon–Krivelevich '21:** $f(\mathbb{Z}_q) = O(q \log q)$,
  $f(\mathbb{Z}_p) \leq 2p$ for p prime.

* **Mészarós–Steiner '22:** $f(A) \leq 8|A|$ for abelian A,
  $f(\mathbb{Z}_p) \leq \frac{3p}{2}$ for p prime.

* **Berendsohn–Boyadzhiyska–Kozma '22:** $f(A) \leq 2|A|$ for any group A.

  They looked for cycles with fixed points in digraphs edge-labeled by functions.

Our results

Our results

Theorem (L.–Morrison '22+). $f(\mathbb{Z}_p) = p + O(\sqrt{p})$.

Our results

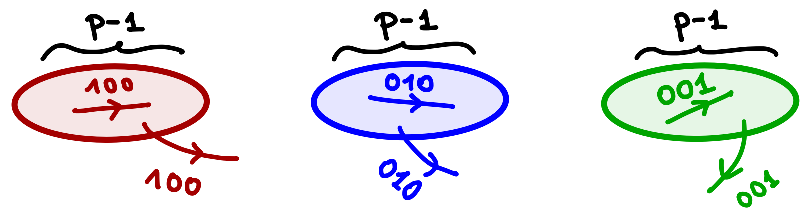
Theorem (L.-Morrison '22+). $f(\mathbb{Z}_p) = p + O(\sqrt{p})$.

$f(\mathbb{Z}_p^k)$

Our results

Theorem (L.–Morrison '22+). $f(\mathbb{Z}_p) = p + O(\sqrt{p})$.

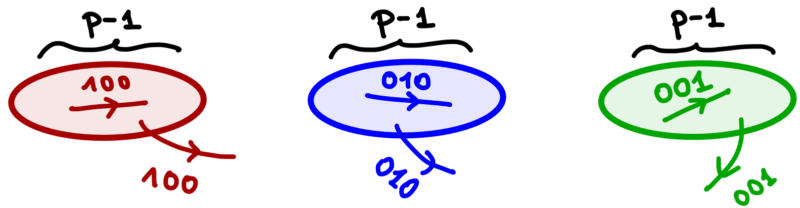
* $f(\mathbb{Z}_p^k) > (p-1)k$:



Our results

Theorem (L.–Morrison '22+). $f(\mathbb{Z}_p) = p + O(\sqrt{p})$.

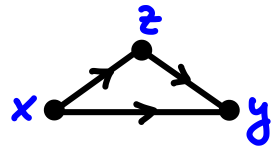
* $f(\mathbb{Z}_p^k) > (p-1)k$:



Theorem (L.–Morrison '22+). $f(\mathbb{Z}_p^k) = O(pk^2 \log k)$ for p prime,

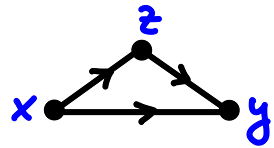
$f(\mathbb{Z}_2^k) = O(k \log k)$.

Definitions

A **gadget** g looks like: 

. Its **value** g^* is $\underbrace{\omega(xzy)}_{\omega(xz)+\omega(yz)} - \omega(xy)$.

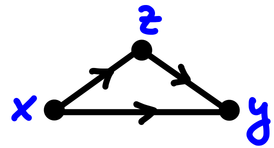
Definitions

A **gadget** g looks like: 

. Its **value** g^* is $\underbrace{\omega(xzy)}_{\omega(xz)+\omega(yz)} - \omega(xy)$.

(Introduced by AK, used also by MS, BBK.)

Definitions

A <u>gadget</u> g looks like: 

. Its <u>value</u> g^* is $\underbrace{\omega(xzy)}_{\omega(xz)+\omega(yz)} - \omega(xy)$.

(Introduced by AK, used also by MS, BBK.)

Given a multisubset $S \subseteq A$, $\Sigma(S) = \left\{ \begin{matrix} \text{subset sums} \\ \text{of } S \end{matrix} \right\} = \left\{ \sum_{t \in T} t \; : \; T \subseteq S \right\}$.

Definitions

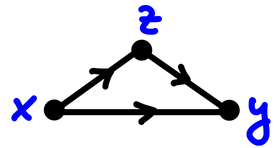
A gadget g looks like:
$$\begin{tikzcd} & z \arrow[dr] & \\ x \arrow[ur] \arrow[rr] & & y \end{tikzcd}$$
. Its value g^* is $\underbrace{\omega(xzy)}_{\omega(xz)+\omega(yz)} - \omega(xy)$.

(Introduced by AK, used also by MS, BBK.)

Given a multisubset $S \subseteq A$, $\Sigma(S) = \left\{ \begin{matrix} \text{subset sums} \\ \text{of } S \end{matrix} \right\} = \left\{ \sum_{t \in T} t \ : \ T \subseteq S \right\}$.

A multisubset $S \subseteq A$ is reduced if $\Sigma(S) \supsetneq \Sigma(S - \{s\})$ $\forall s \in S$.

Definitions

A gadget g looks like: 

. Its value g^* is $\underbrace{\omega(xzy)}_{\omega(xz)+\omega(yz)} - \omega(xy)$.

(Introduced by AK, used also by MS, BBK.)

Given a multisubset $S \subseteq A$, $\Sigma(S) = \left\{ \begin{matrix}\text{subset sums}\\ \text{of } S\end{matrix} \right\} = \left\{ \sum_{t \in T} t : T \subseteq S \right\}$.

A multisubset $S \subseteq A$ is reduced if $\Sigma(S) \supsetneq \Sigma(S - \{s\})$ $\forall s \in S$.

Define $h_p(k) = \max\{|S| : S \text{ reduced multisubset in } \mathbb{Z}_p^k\}$.

Using gadgets

Observation. $\omega: E(\vec{K}_n) \to A$. Let $\mathcal{G}$ be a collection of disjoint gadgets, satisfying $\Sigma(\mathcal{G}^*) = A$. Then there is a zero-sum cycle.

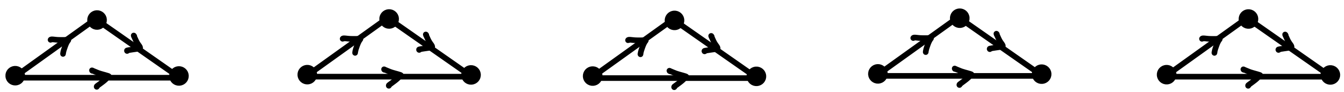
$\{g^* : g \in \mathcal{G}\}$

Using gadgets

Observation. $\omega : E(\vec{K}_n) \to A$. Let $\mathcal{G}$ be a collection of disjoint gadgets, satisfying $\Sigma(\mathcal{G}^*) = A$. Then there is a zero-sum cycle.

$\{g^* : g \in \mathcal{G}\}$

Proof.

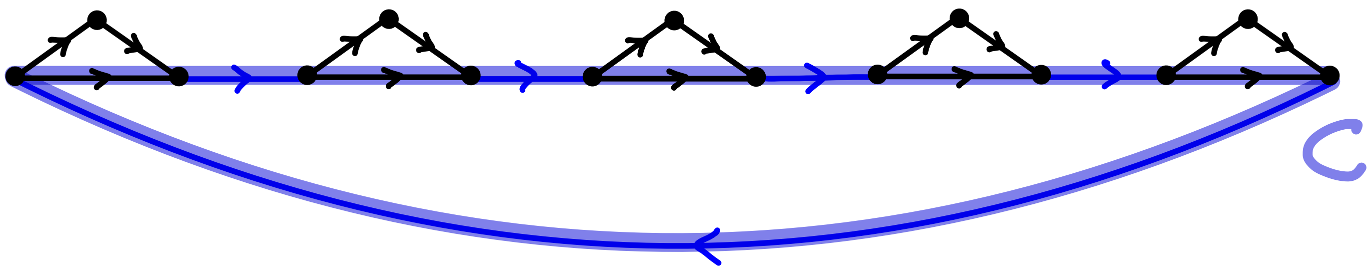


Using gadgets

Observation. $\omega : E(\vec{K}_n) \longrightarrow A$. Let $\mathcal{G}$ be a collection of disjoint gadgets, satisfying $\Sigma(\mathcal{G}^*) = A$. Then there is a zero-sum cycle.

$\{g^* : g \in \mathcal{G}\}$

Proof.

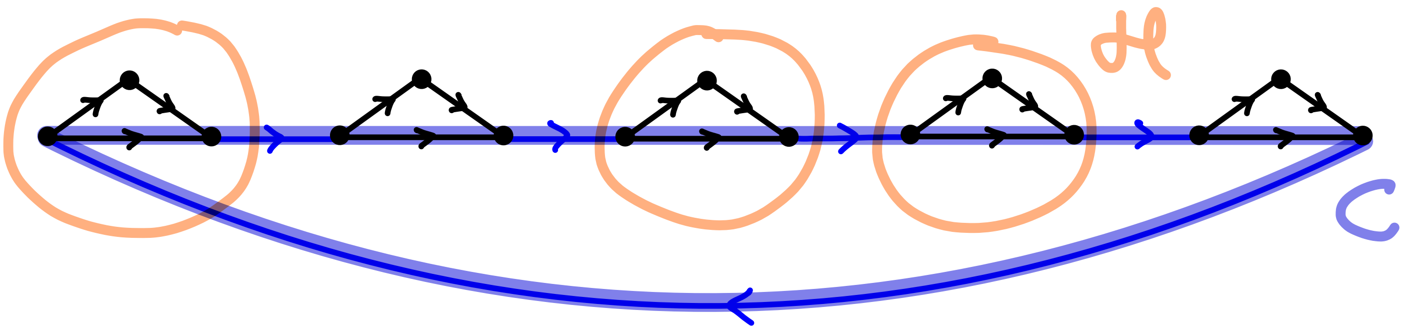


Using gadgets

Observation. $\omega : E(\vec{K}_n) \to A$. Let $\mathcal{G}$ be a collection of disjoint gadgets, satisfying $\Sigma(\mathcal{G}^*) = A$. Then there is a zero-sum cycle.

$\{g^* : g \in \mathcal{G}\}$

Proof.



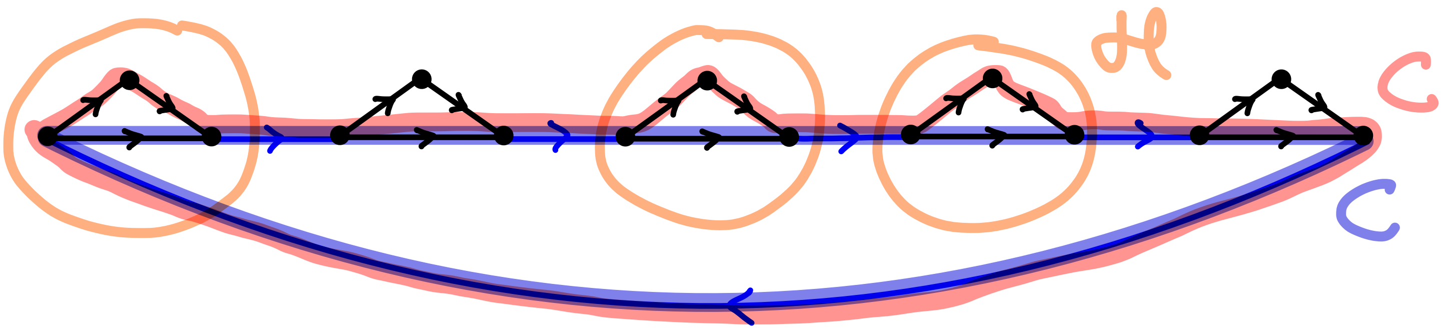
$\Sigma(\mathcal{G}^*) = A \Rightarrow \exists \mathcal{H} \subseteq \mathcal{G} : \sum_{h \in \mathcal{H}} h^* = -\omega(C)$.

Using gadgets

Observation. $\omega : E(\vec{K}_n) \to A$. Let $\mathcal{G}$ be a collection of disjoint gadgets, satisfying $\Sigma(\mathcal{G}^*) = A$. Then there is a zero-sum cycle.

$\{g^* : g \in \mathcal{G}\}$

Proof.



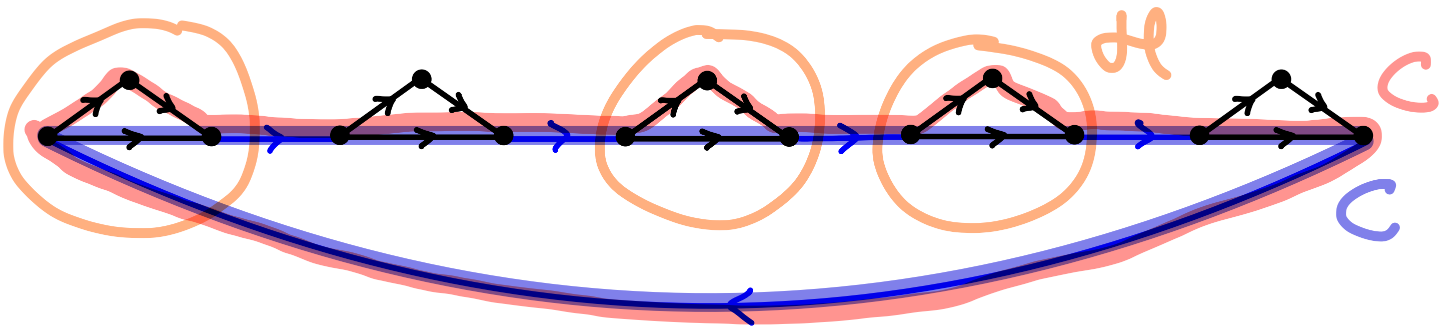
$\Sigma(\mathcal{G}^*) = A \Rightarrow \exists \mathcal{H} \subseteq \mathcal{G} : \sum_{h \in \mathcal{H}} h^* = -\omega(C)$.

Using gadgets

Observation. $\omega : E(\vec{K}_n) \to A$. Let $\mathcal{G}$ be a collection of disjoint gadgets, satisfying $\Sigma(\mathcal{G}^*) = A$. Then there is a zero-sum cycle.

$\{g^* : g \in \mathcal{G}\}$

Proof.



$\Sigma(\mathcal{G}^*) = A \Rightarrow \exists \mathcal{H} \subseteq \mathcal{G} : \sum_{h \in \mathcal{H}} h^* = -\omega(C)$.

$\omega(C') = \omega(C) + \sum_{h \in \mathcal{H}} h^* = 0$. $\square$

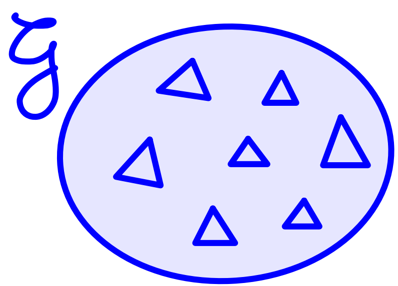
Bounding $f(\mathbb{Z}_p^k)$

Suppose: $\omega : E(\vec{K}_n) \longrightarrow \mathbb{Z}_p^k$ has no zero-sum cycles.

Bounding $f(\mathbb{Z}_p^k)$

Suppose: $\omega : E(\vec{K}_n) \to \mathbb{Z}_p^k$ has no zero-sum cycles.

Pick $\mathcal{G}$ family of disjoint gadgets with max $|\mathcal{E}(\mathcal{G}^*)|$ & min $|\mathcal{G}|$.

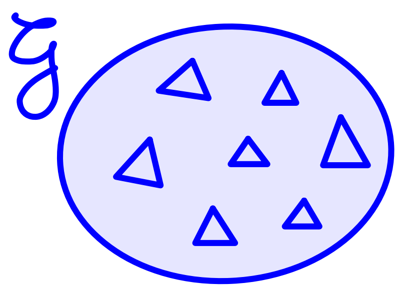


Bounding $f(\mathbb{Z}_p^k)$

Suppose: $\omega : E(\vec{K}_n) \to \mathbb{Z}_p^k$ has no zero-sum cycles.

Pick $\mathcal{G}$ family of disjoint gadgets with max $|\Sigma(\mathcal{G}^*)|$ & min $|\mathcal{G}|$.

$\Rightarrow$ $\mathcal{G}^*$ is reduced $\Rightarrow$ $|\mathcal{G}| \leq h_p(k)$.



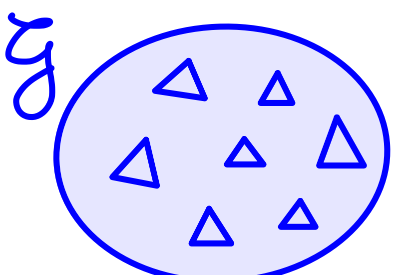
Bounding $f(\mathbb{Z}_p^k)$

Suppose: $\omega : E(\vec{K}_n) \to \mathbb{Z}_p^k$ has no zero-sum cycles.

Pick $\mathcal{G}$ family of disjoint gadgets with max $|\Sigma(\mathcal{G}^*)|$ & min $|\mathcal{G}|$.

$\Rightarrow$ $\mathcal{G}^*$ is reduced $\Rightarrow$ $|\mathcal{G}| \leq h_p(k)$.

Observation $\Rightarrow$ $\Sigma(\mathcal{G}^*) \neq \mathbb{Z}_p^k$.



Bounding $f(\mathbb{Z}_p^k)$

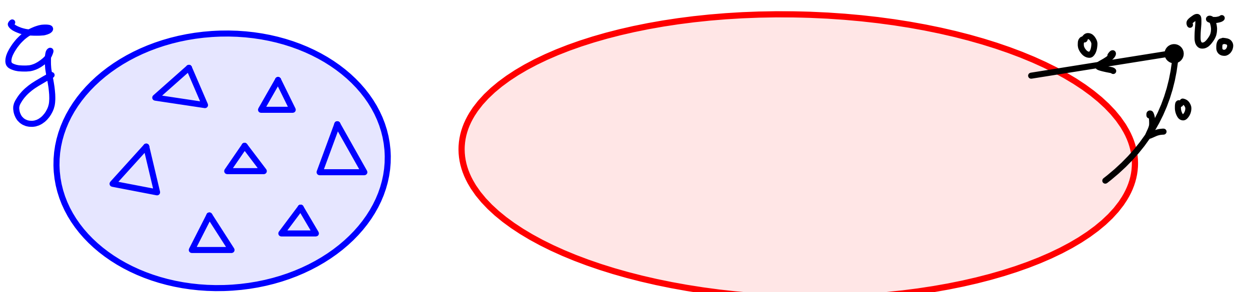
Suppose: $\omega : E(\vec{K}_n) \to \mathbb{Z}_p^k$ has no zero-sum cycles.

Pick $\mathcal{G}$ family of disjoint gadgets with max $|\Sigma(\mathcal{G}^*)|$ & min $|\mathcal{G}|$.

$\Rightarrow$ $\mathcal{G}^*$ is reduced $\Rightarrow$ $|\mathcal{G}| \leq h_p(k)$.

Observation $\Rightarrow$ $\Sigma(\mathcal{G}^*) \neq \mathbb{Z}_p^k$.

Claim. Let $v_0 \notin V(\mathcal{G})$. We may assume $\omega(v_0 u) = 0$ $\forall u \neq v_0$.



Bounding $f(\mathbb{Z}_p^k)$

Define: $B = \{ b \in \mathbb{Z}_p^k : b + \Sigma(\mathcal{G}^*) = \Sigma(\mathcal{G}^*) \}$.

Bounding $f(\mathbb{Z}_p^k)$

Define: $B = \{ b \in \mathbb{Z}_p^k : b + \Sigma(\mathcal{G}^*) = \Sigma(\mathcal{G}^*) \}$.

* B subgroup.
* $B \subseteq \Sigma(\mathcal{G}^*)$.

Bounding $f(\mathbb{Z}_p^k)$

Define: $B = \{ b \in \mathbb{Z}_p^k : b + \Sigma(\mathcal{G}^*) = \Sigma(\mathcal{G}^*) \}$.

$$\left.\begin{array}{l} * \ \ B \text{ subgroup.} \\ * \ \ B \subseteq \Sigma(\mathcal{G}^*). \end{array}\right\} \Longrightarrow B \cong \mathbb{Z}_p^{k'},\ k' < k.$$

Bounding $f(\mathbb{Z}_p^k)$

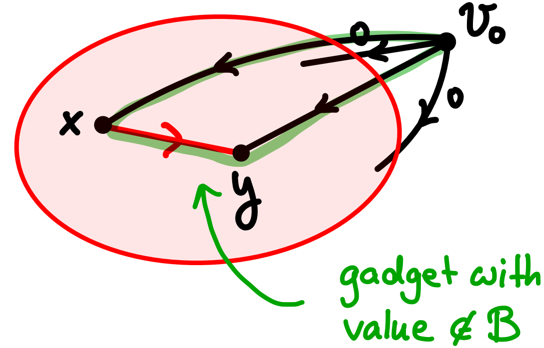
Define: $B = \{ b \in \mathbb{Z}_p^k : b + \Sigma(\mathcal{G}^*) = \Sigma(\mathcal{G}^*) \}$.

$$\left.\begin{array}{l} * \ B \text{ subgroup.} \\ * \ B \subseteq \Sigma(\mathcal{G}^*). \end{array}\right\} \Longrightarrow B \cong \mathbb{Z}_p^{k'},\ k' < k.$$

* $\forall x, y \notin V(\mathcal{G}) \cup \{v_0\}$: $\omega(xy) \in B$.

Bounding $f(\mathbb{Z}_p^k)$

Define: $B = \{ b \in \mathbb{Z}_p^k : b + \Sigma(\mathcal{G}^*) = \Sigma(\mathcal{G}^*) \}$.



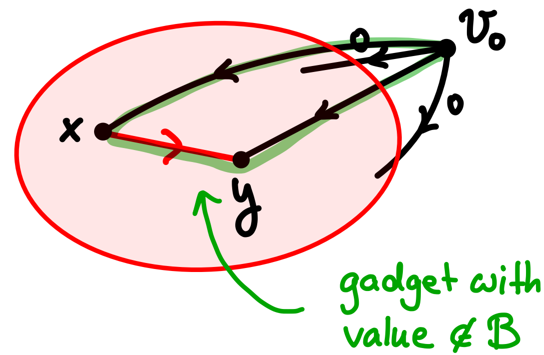
* B subgroup.
* $B \subseteq \Sigma(\mathcal{G}^*)$.

$\Big\} \Longrightarrow B \cong \mathbb{Z}_p^{k'}$, $k' < k$.

* $\forall x, y \notin V(\mathcal{G}) \cup \{v_0\}$: $\omega(xy) \in B$.

(Otherwise $\mathcal{G} \cup \{\text{gadget on } x, y, v_0\}$ contradicts maximality of $\Sigma\mathcal{G}^*$.)

Bounding $f(\mathbb{Z}_p^k)$



Define: $B = \{ b \in \mathbb{Z}_p^k : b + \Sigma(\mathcal{G}^*) = \Sigma(\mathcal{G}^*) \}$.

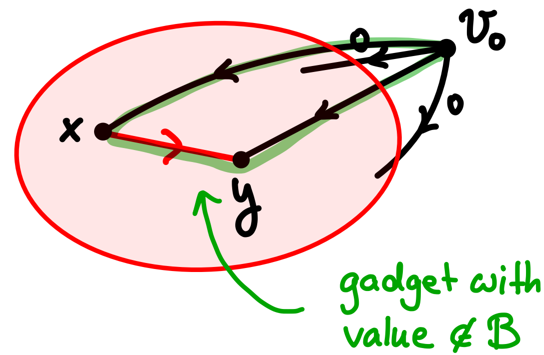
$$\left.\begin{array}{l} * \ B \text{ subgroup.} \\ * \ B \subseteq \Sigma(\mathcal{G}^*). \end{array}\right\} \Longrightarrow B \cong \mathbb{Z}_p^{k'},\ k' < k.$$

* $\forall x, y \notin V(\mathcal{G}) \cup \{v_0\}$: $\omega(xy) \in B$.

(Otherwise $\mathcal{G} \cup \{\text{gadget on } x, y, v_0\}$ contradicts maximality of $\Sigma\mathcal{G}^*$.)

Induction on $(V(\mathcal{G}) \cup \{v_0\})^c$ gives: $f(\mathbb{Z}_p^k) = O(k \cdot h_p(k))$.

Bounding $f(\mathbb{Z}_p^k)$



Define: $B = \{ b \in \mathbb{Z}_p^k : b + \Sigma(\mathcal{G}^*) = \Sigma(\mathcal{G}^*) \}$.

$\left.\begin{array}{l} * \ B \text{ subgroup.} \\ * \ B \subseteq \Sigma(\mathcal{G}^*). \end{array}\right\} \Longrightarrow B \cong \mathbb{Z}_p^{k'},\ k' < k.$

* $\forall x, y \notin V(\mathcal{G}) \cup \{v_0\}:\ \omega(xy) \in B$.

(Otherwise $\mathcal{G} \cup \{\text{gadget on } x, y, v_0\}$ contradicts maximality of $\Sigma \mathcal{G}^*$.)

Induction on $(V(\mathcal{G}) \cup \{v_0\})^c$ gives: $f(\mathbb{Z}_p^k) = O(k \cdot h_p(k))$.

Lemma. $f(\mathbb{Z}_p^k) = O(\log k \cdot h_p(k))$.

Easy facts about reduced sets

Recall: $S \subseteq A$ is reduced if $\Sigma(S) \supsetneq \Sigma(S - \{s\})$ $\forall s \in S$.

$h_p(k)$ = max size of a reduced multisubset of $\mathbb{Z}_p^k$.

Easy facts about reduced sets

Recall: $S \subseteq A$ is reduced if $\Sigma(S) \supsetneq \Sigma(S - \{s\})$ $\forall s \in S$.

$h_p(k)$ = max size of a reduced multisubset of $\mathbb{Z}_p^k$.

* $S \subseteq \mathbb{Z}_2^k$ is reduced iff it is linearly independent $\Rightarrow$ $h_2(k) = k$.

Easy facts about reduced sets

Recall: $S \subseteq A$ is <u>reduced</u> if $\Sigma(S) \supsetneq \Sigma(S - \{s\})$ $\forall s \in S$.

$h_p(k)$ = max size of a reduced multisubset of $\mathbb{Z}_p^k$.

* $S \subseteq \mathbb{Z}_2^k$ is reduced iff it is linearly independent $\Rightarrow$ $h_2(k) = k$.

* $h_p(k) \geq (p-1)k$: $\underbrace{100, \ldots, 100}_{p-1}, \underbrace{010, \ldots, 010}_{p-1}, \underbrace{001, \ldots, 001}_{p-1}$.

Easy facts about reduced sets

Recall: $S \subseteq A$ is <u>reduced</u> if $\Sigma(S) \supsetneq \Sigma(S - \{s\})$ $\forall s \in S$.

$h_p(k)$ = max size of a reduced multisubset of $\mathbb{Z}_p^k$.

* $S \subseteq \mathbb{Z}_2^k$ is reduced iff it is linearly independent $\Rightarrow$ $h_2(k) = k$.

* $h_p(k) \geq (p-1)k$: $\underbrace{100, \ldots, 100}_{p-1}, \underbrace{010, \ldots, 010}_{p-1}, \underbrace{001, \ldots, 001}_{p-1}$.

* $h_p(1) = p-1$.

Easy facts about reduced sets

Recall: $S \subseteq A$ is reduced if $\Sigma(S) \supsetneq \Sigma(S - \{s\})$ $\forall s \in S$.

$h_p(k)$ = max size of a reduced multisubset of $\mathbb{Z}_p^k$.

* $S \subseteq \mathbb{Z}_2^k$ is reduced iff it is linearly independent $\Rightarrow$ $h_2(k) = k$.

* $h_p(k) \geq (p-1)k$: $\overbrace{100, \ldots, 100}^{p-1}, \overbrace{010, \ldots, 010}^{p-1}, \overbrace{001, \ldots, 001}^{p-1}$.

* $h_p(1) = p-1$.

notice: S reduced & $S' \subseteq S$ $\Rightarrow$ S' reduced.

Easy facts about reduced sets

Recall: $S \subseteq A$ is <u>reduced</u> if $\Sigma(S) \supsetneq \Sigma(S-\{s\})$ $\forall s \in S$.

$h_p(k)$ = max size of a reduced multisubset of $\mathbb{Z}_p^k$.

* $S \subseteq \mathbb{Z}_2^k$ is reduced iff it is linearly independent $\Rightarrow$ $h_2(k)=k$.

* $h_p(k) \geq (p-1)k$: $\underbrace{100,\ldots,100}_{p-1}, \underbrace{010,\ldots,010}_{p-1}, \underbrace{001,\ldots,001}_{p-1}$.

* $h_p(1) = p-1$.

notice: S reduced & $S' \subseteq S$ $\Rightarrow$ S' reduced.

So: if $S=\{s_1,\ldots,s_k\}$ is reduced then:

$$|\Sigma S| = |\Sigma(\{s_1,\ldots,s_k\})| \geq |\Sigma(\{s_1,\ldots,s_{k-1}\})| + 1 \geq \ldots \geq |\Sigma(\emptyset)| + k \geq k+1.$$

Easy facts about reduced sets

Recall: $S \subseteq A$ is reduced if $\Sigma(S) \supsetneq \Sigma(S-\{s\})$ $\forall s \in S$.

$h_p(k)$ = max size of a reduced multisubset of $\mathbb{Z}_p^k$.

* $S \subseteq \mathbb{Z}_2^k$ is reduced iff it is linearly independent $\Rightarrow$ $h_2(k)=k$.

* $h_p(k) \geq (p-1)k$: $\underbrace{100,\dots,100}_{p-1}, \underbrace{010,\dots,010}_{p-1}, \underbrace{001,\dots,001}_{p-1}$.

* $h_p(1)=p-1$.

notice: S reduced & $S' \subseteq S$ $\Rightarrow$ S' reduced.

So: if $S=\{s_1,\dots,s_k\}$ is reduced then:

$$|\Sigma S| = |\Sigma(\{s_1,\dots,s_k\})| \geq |\Sigma(\{s_1,\dots,s_{k-1}\})| + 1 \geq \dots \geq |\Sigma(\emptyset)| + k \geq k+1.$$

$\Rightarrow$ if $S \subseteq \mathbb{Z}_p$ is reduced then $|S| \leq p-1$.)

An upper bound

Theorem (L.-Morrison '22+). $h_p(k) \leq (p-1)\binom{k+1}{2}$.

An upper bound

Theorem (L.–Morrison '22+). $h_p(k) \leq (p-1)\binom{k+1}{2}$.

Today: $k=2$.

An upper bound

Theorem (L.–Morrison '22+). $h_p(k) \leq (p-1)\binom{k+1}{2}$.

Today: $k=2$.

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

An upper bound

Theorem (L.-Morrison '22+). $h_p(k) \leq (p-1)\binom{k+1}{2}$.

Today: $k=2$.

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

Lemma $\Rightarrow$ Theorem for $k=2$:

An upper bound

Theorem (L.–Morrison '22+). $h_p(k) \leq (p-1)\binom{k+1}{2}$.

Today: $k=2$.

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

Lemma $\Rightarrow$ Theorem for $k=2$:

Suppose: S reduced multisubset of $\mathbb{Z}_p^2$ of size $> 3(p-1)$.

An upper bound

Theorem (L.–Morrison '22+). $h_p(k) \le (p-1)\binom{k+1}{2}$.

Today: $k=2$.

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\le p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

Lemma $\Rightarrow$ Theorem for $k=2$:

Suppose: S reduced multisubset of $\mathbb{Z}_p^2$ of size $> 3(p-1)$.

By $h_p(1) = p-1$, S has $\le p-1$ vectors in each direction.

An upper bound

Theorem (L.-Morrison '22+). $h_p(k) \leq (p-1)\binom{k+1}{2}$.

Today: $k=2$.

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma_1(S)=\mathbb{Z}_p^2$.

Lemma $\Rightarrow$ Theorem for $k=2$:

Suppose: S reduced multisubset of $\mathbb{Z}_p^2$ of size $> 3(p-1)$.

By $h_p(1)=p-1$, S has $\leq p-1$ vectors in each direction.

Lemma $\Rightarrow$ if $S' \subseteq S$ has size $3(p-1)$ then $\Sigma_1(S')=\mathbb{Z}_p^2$

An upper bound

Theorem (L.–Morrison '22+). $h_p(k) \leq (p-1)\binom{k+1}{2}$.

Today: $k=2$.

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

Lemma $\Rightarrow$ Theorem for $k=2$:

Suppose: S reduced multisubset of $\mathbb{Z}_p^2$ of size $> 3(p-1)$.

By $h_p(1) = p-1$, S has $\leq p-1$ vectors in each direction.

Lemma $\Rightarrow$ if $S' \subseteq S$ has size $3(p-1)$ then $\Sigma(S') = \mathbb{Z}_p^2$

※ to $\Sigma(S') \subsetneq \Sigma(S)$.

Proof of Lemma

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

Proof of Lemma

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

Let $\omega = \binom{c}{d} \in \mathbb{Z}_p^2$. **Aim:** show $\omega \in \Sigma(S)$.

Proof of Lemma

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S)=\mathbb{Z}_p^2$.

Let $\omega=\binom{c}{d}\in\mathbb{Z}_p^2$. **Aim:** show $\omega\in\Sigma(S)$.

Write $m=2(p-1)$.

Define: $P(x_1,\_\_,x_m)=$

$$\left(1-(x_1a_1+\_\_+x_ma_m-c)^{p-1}\right)\cdot\left(1-(x_1b_1+\_\_+x_mb_m-d)^{p-1}\right).$$

Proof of Lemma

Lemma. Let S be a multisubset of $\mathbb{Z}_p^2$ of size $3(p-1)$ with $\leq p-1$ vectors in each direction. Then $\Sigma(S) = \mathbb{Z}_p^2$.

Let $\omega = \binom{c}{d} \in \mathbb{Z}_p^2$. **Aim:** show $\omega \in \Sigma(S)$.

Write $m = 2(p-1)$.

Define: $P(x_1, \_\_, x_m) =$

$$\left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right).$$

Notice: $P(x_1, \_\_, x_m) \neq 0$ for some $x_1, \_\_, x_m \in \{0,1\}$ iff $\omega \in \Sigma\left(\left\{\binom{a_1}{b_1}, \_\_, \binom{a_m}{b_m}\right\}\right)$.

$m = 2(p-1)$.

$$P(x_1, \ldots, x_m) = \left(1 - (x_1 a_1 + \ldots + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \ldots + x_m b_m - d)^{p-1}\right)$$

$P(x_1, \ldots, x_m) \neq 0$ for some $x_1, \ldots, x_m \in \{0,1\}$ iff $\omega \in \Sigma\left(\left\{\binom{a_1}{b_1}, \ldots, \binom{a_m}{b_m}\right\}\right)$.

$m = 2(p-1)$.

$$P(x_1, \_\_, x_m) = \left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right)$$

$P(x_1, \_\_, x_m) \neq 0$ for some $x_1, \_\_, x_m \in \{0,1\}$ iff $\omega \in \Sigma\left(\left\{\binom{a_1}{b_1}, \_\_, \binom{a_m}{b_m}\right\}\right)$.

<u>Combinatorial Nullstellensatz (Alon '99)</u>. Let $f \in \mathbb{F}[x_1, \_\_, x_n]$ be a polynomial of degree $\sum_i t_i$ s.t. the coefficient of $\prod_i x_i^{t_i}$ is $\neq 0$. Let $S_1, \_\_, S_n \subseteq \mathbb{F}$ satisfy $|S_i| > t_i$. Then $\exists s_i \in S_i$: $f(s_1, \_\_, s_n) \neq 0$.

$m = 2(p-1)$.

$$P(x_1, \_\_, x_m) = \left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right)$$

$P(x_1, \_\_, x_m) \neq 0$ for some $x_1, \_\_, x_m \in \{0,1\}$ iff $\omega \in \Sigma\left(\left\{\binom{a_1}{b_1}, \_\_, \binom{a_m}{b_m}\right\}\right)$.

Combinatorial Nullstellensatz (Alon '99). Let $f \in \mathbb{F}[x_1, \_\_, x_n]$ be a polynomial of degree $\sum_i t_i$ s.t. the coefficient of $\prod_i x_i^{t_i}$ is $\neq 0$. Let $S_1, \_\_, S_n \subseteq \mathbb{F}$ satisfy $|S_i| > t_i$. Then $\exists s_i \in S_i$: $f(s_1, \_\_, s_n) \neq 0$.

Taking $S_i = \{0,1\}$ & $t_i = 1$, we get:

$m = 2(p-1)$.

$$P(x_1, \_\_, x_m) = \left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right)$$

$P(x_1, \_\_, x_m) \neq 0$ for some $x_1, \_\_, x_m \in \{0,1\}$ iff $\omega \in \Sigma\left(\left\{\binom{a_1}{b_1}, \_\_, \binom{a_m}{b_m}\right\}\right)$.

Combinatorial Nullstellensatz (Alon '99). Let $f \in \mathbb{F}[x_1, \_\_, x_n]$ be a polynomial of degree $\sum_i t_i$ s.t. the coefficient of $\prod_i x_i^{t_i}$ is $\neq 0$. Let $S_1, \_\_, S_n \subseteq \mathbb{F}$ satisfy $|S_i| > t_i$. Then $\exists s_i \in S_i$: $f(s_1, \_\_, s_n) \neq 0$.

Taking $S_i = \{0,1\}$ & $t_i = 1$, we get:

if $\underbrace{[x_1 \cdot \_\_ \cdot x_m] P(x_1, \_\_, x_m)}_{\text{the coefficient of } x_1 \cdot \_\_ \cdot x_m \text{ in } P(x_1, \_\_, x_m)} \neq 0$ then $\omega \in \Sigma\left(\left\{\binom{a_1}{b_1}, \_\_, \binom{a_m}{b_m}\right\}\right)$.

$m = 2(p-1)$,

$$P(x_1, \_\_, x_m) = \left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right).$$

$m=2(p-1)$,

$$P(x_1,\_\_,x_m)=\left(1-(x_1a_1+\_\_+x_ma_m-c)^{p-1}\right)\cdot\left(1-(x_1b_1+\_\_+x_mb_m-d)^{p-1}\right).$$

$$[x_1\cdot\_\_\cdot x_m]P(x_1,\_\_,x_m)$$

$$=[x_1\cdot\_\_\cdot x_m]\left((x_1a_1+\_\_+x_ma_m)^{p-1}(x_1b_1+\_\_+x_mb_m)^{p-1}\right)$$

$m = 2(p-1)$,

$$P(x_1, \_\_, x_m) = \left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right).$$

$$[x_1 \cdot \_\_ \cdot x_m] P(x_1, \_\_, x_m)$$

$$= [x_1 \cdot \_\_ \cdot x_m] \left((x_1 a_1 + \_\_ + x_m a_m)^{p-1} (x_1 b_1 + \_\_ + x_m b_m)^{p-1}\right)$$

$$= \sum_{\substack{I \,\dot\cup\, J = [m] \\ |I| = |J|}} \left((p-1)!\right)^2 \prod_{i \in I} a_i \prod_{j \in J} b_j$$

$m=2(p-1)$,

$$P(x_1,\_\_,x_m)=\left(1-(x_1a_1+\_\_+x_ma_m-c)^{p-1}\right)\cdot\left(1-(x_1b_1+\_\_+x_mb_m-d)^{p-1}\right).$$

$$[x_1\cdot\_\_\cdot x_m]P(x_1,\_\_,x_m)$$

$$=[x_1\cdot\_\_\cdot x_m]\left((x_1a_1+\_\_+x_ma_m)^{p-1}(x_1b_1+\_\_+x_mb_m)^{p-1}\right)$$

$$=\sum_{\substack{I\dot\cup J=[m]\\|I|=|J|}}\underbrace{((p-1)!)^2}_{=1}\prod_{i\in I}a_i\prod_{j\in J}b_j \quad =\sum_{\substack{I\dot\cup J=[m]\\|I|=|J|}}\prod_{i\in I}a_i\prod_{j\in J}b_j.$$

$m = 2(p-1)$,

$$P(x_1, \_\_, x_m) = \left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right).$$

$$[x_1 \cdot \_\_ \cdot x_m] P(x_1, \_\_, x_m)$$

$$= [x_1 \cdot \_\_ \cdot x_m] \left((x_1 a_1 + \_\_ + x_m a_m)^{p-1} (x_1 b_1 + \_\_ + x_m b_m)^{p-1} \right)$$

$$= \sum_{\substack{I \,\dot\cup\, J = [m] \\ |I| = |J|}} \underbrace{\left((p-1)!\right)^2}_{=1} \prod_{i \in I} a_i \prod_{j \in J} b_j \quad = \sum_{\substack{I \,\dot\cup\, J = [m] \\ |I| = |J|}} \prod_{i \in I} a_i \prod_{j \in J} b_j .$$

Claim. $\exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\displaystyle\sum_{\substack{I \,\dot\cup\, J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j \neq 0$ for $0 \leq \ell \leq p-1$.

$m = 2(p-1)$,

$$P(x_1, \_\_, x_m) = \left(1 - (x_1 a_1 + \_\_ + x_m a_m - c)^{p-1}\right) \cdot \left(1 - (x_1 b_1 + \_\_ + x_m b_m - d)^{p-1}\right).$$

$$[x_1 \cdot \_\_ \cdot x_m] P(x_1, \_\_, x_m)$$

$$= [x_1 \cdot \_\_ \cdot x_m] \left((x_1 a_1 + \_\_ + x_m a_m)^{p-1} (x_1 b_1 + \_\_ + x_m b_m)^{p-1} \right)$$

$$= \sum_{\substack{I \cup J = [m] \\ |I| = |J|}} \underbrace{((p-1)!)^2}_{=1} \prod_{i \in I} a_i \prod_{j \in J} b_j \quad = \sum_{\substack{I \cup J = [m] \\ |I| = |J|}} \prod_{i \in I} a_i \prod_{j \in J} b_j .$$

<u>Claim</u>. $\exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\sum_{\substack{I \cup J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j \neq 0$ for $0 \leq \ell \leq p-1$.

Claim with $\ell = p-1 \Rightarrow \Sigma(S) = \mathbb{Z}_p^2$.

Claim. $\exists \left\{ \binom{a_1}{b_1}, \_\_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\underbrace{\sum_{\substack{I \,\dot\cup\, J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j \neq 0}_{(\Delta)_\ell}$ for $0 \le \ell \le p-1$.

Proof of claim.

Claim. $\exists \left\{ \binom{a_1}{b_1}, \_\_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\underbrace{\sum_{\substack{I \,\dot\cup\, J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j}_{(\Delta)_\ell} \neq 0$ for $0 \leq \ell \leq p-1$.

Proof of claim.

$\underline{\ell = 0}$: $(\Delta)_0 = \prod_{i \in [p-1]} a_i$

Claim. $\exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\underbrace{\sum_{\substack{I \cup J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j}_{(\Delta)_\ell} \neq 0$ for $0 \leq \ell \leq p-1$.

Proof of claim.

$\underline{\ell = 0}$: $(\Delta)_0 = \prod_{i \in [p-1]} a_i$ & S has $\leq p-1$ elements $\binom{0}{b}$ $\Rightarrow \exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1}}{b_{p-1}} \right\} \subseteq S$: $(\Delta)_0 \neq 0$.

Claim. $\exists \left\{ \binom{a_1}{b_1}, \ldots, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\underbrace{\sum_{\substack{I \cup J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j}_{(\Delta)_\ell} \neq 0$ for $0 \le \ell \le p-1$.

Proof of claim.

$\ell = 0$: $(\Delta)_0 = \prod_{i \in [p-1]} a_i$ & S has $\le p-1$ elements $\binom{0}{b}$ $\Rightarrow \exists \left\{ \binom{a_1}{b_1}, \ldots, \binom{a_{p-1}}{b_{p-1}} \right\} \subseteq S$: $(\Delta)_0 \neq 0$.

$\ell \to \ell+1$: by induction $\exists \left\{ \binom{a_1}{b_1}, \ldots, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $(\Delta)_\ell \neq 0$.

Claim. $\exists\left\{\binom{a_1}{b_1}, \_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}}\right\} \subseteq S$ s.t. $\underbrace{\sum_{\substack{I \uplus J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j}_{(\Delta)_\ell} \neq 0$ for $0 \leq \ell \leq p-1$.

Proof of claim.

$\ell = 0$: $(\Delta)_0 = \prod_{i \in [p-1]} a_i$ & S has $\leq p-1$ elements $\binom{0}{b} \Rightarrow \exists\left\{\binom{a_1}{b_1}, \_, \binom{a_{p-1}}{b_{p-1}}\right\} \subseteq S$: $(\Delta)_0 \neq 0$.

$\ell \to \ell+1$: by induction $\exists\left\{\binom{a_1}{b_1}, \_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}}\right\} \subseteq S$ s.t. $(\Delta)_\ell \neq 0$.

Notice: $(\Delta)_{\ell+1} = (\square)\, a_{p+\ell} + (\Delta)_\ell\, b_{p+\ell}$.

Claim. $\exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\underbrace{\sum_{\substack{I \cup J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j}_{(\Delta)_\ell} \neq 0$ for $0 \leq \ell \leq p-1$.

Proof of claim.

$\underline{\ell = 0}$: $(\Delta)_0 = \prod_{i \in [p-1]} a_i$ & S has $\leq p-1$ elements $\binom{0}{b} \Rightarrow \exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1}}{b_{p-1}} \right\} \subseteq S$: $(\Delta)_0 \neq 0$.

$\underline{\ell \to \ell+1}$: by induction $\exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $(\Delta)_\ell \neq 0$.

Notice: $(\Delta)_{\ell+1} = (\square)\, a_{p+\ell} + (\Delta)_\ell\, b_{p+\ell}$.

S has $\leq p-1$ vectors $\binom{a}{b}$ s.t. $a \cdot (\square) + b \cdot (\Delta)_\ell = 0$

Claim. $\exists\left\{\binom{a_1}{b_1},\ldots,\binom{a_{p-1+\ell}}{b_{p-1+\ell}}\right\}\subseteq S$ s.t. $\underbrace{\sum_{\substack{I\cup J=[p-1+\ell]\\|I|=p-1}}\prod_{i\in I}a_i\prod_{j\in J}b_j}_{(\Delta)_\ell}\neq 0$ for $0\leq\ell\leq p-1$.

Proof of claim.

$\ell=0$: $(\Delta)_0=\prod_{i\in[p-1]}a_i$ & S has $\leq p-1$ elements $\binom{0}{b}\Rightarrow\exists\left\{\binom{a_1}{b_1},\ldots,\binom{a_{p-1}}{b_{p-1}}\right\}\subseteq S$: $(\Delta)_0\neq 0$.

$\ell\to\ell+1$: by induction $\exists\left\{\binom{a_1}{b_1},\ldots,\binom{a_{p-1+\ell}}{b_{p-1+\ell}}\right\}\subseteq S$ s.t. $(\Delta)_\ell\neq 0$.

Notice: $(\Delta)_{\ell+1}=(\square)\,a_{p+\ell}+(\Delta)_\ell\, b_{p+\ell}$.

S has $\leq p-1$ vectors $\binom{a}{b}$ s.t. $a\cdot(\square)+b\cdot(\Delta)_\ell=0$

$\leq 2(p-1)-1$ vectors are used

Claim. $\exists \left\{ \binom{a_1}{b_1}, \_\_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $\underbrace{\sum_{\substack{I \cup J = [p-1+\ell] \\ |I| = p-1}} \prod_{i \in I} a_i \prod_{j \in J} b_j}_{(\Delta)_\ell} \neq 0$ for $0 \leq \ell \leq p-1$.

Proof of claim.

$\underline{\ell = 0}$: $(\Delta)_0 = \prod_{i \in [p-1]} a_i$ & S has $\leq p-1$ elements $\binom{0}{b}$ $\Rightarrow \exists \left\{ \binom{a_1}{b_1}, \_, \binom{a_{p-1}}{b_{p-1}} \right\} \subseteq S$: $(\Delta)_0 \neq 0$.

$\underline{\ell \to \ell+1}$: by induction $\exists \left\{ \binom{a_1}{b_1}, \_, \binom{a_{p-1+\ell}}{b_{p-1+\ell}} \right\} \subseteq S$ s.t. $(\Delta)_\ell \neq 0$.

Notice: $(\Delta)_{\ell+1} = (\square)\, a_{p+\ell} + (\Delta)_\ell\, b_{p+\ell}$.

S has $\leq p-1$ vectors $\binom{a}{b}$ s.t. $a \cdot (\square) + b \cdot (\Delta)_\ell = 0$

$\leq 2(p-1)-1$ vectors are used

$\Rightarrow$ suitable $\binom{a_{p+\ell}}{b_{p+\ell}}$ exists. $\square$

Open problems

Open problems

* We know: $h_p(k) \geq k(p-1)$ and showed: $h_p(k) \leq \binom{k+1}{2}(p-1)$.

  Is $h_p(k) = k(p-1)$?

Open problems

* We know: $h_p(k) \geq k(p-1)$ and showed: $h_p(k) \leq \binom{k+1}{2}(p-1)$.

  Is $h_p(k) = k(p-1)$?

  In particular, is $h_p(2) = 2(p-1)$? We can show: $h_p(2) \lesssim 2.5p$.

Open problems

* We know: $h_p(k) \geq k(p-1)$ and showed: $h_p(k) \leq \binom{k+1}{2}(p-1)$.

  Is $h_p(k) = k(p-1)$?

  In particular, is $h_p(2) = 2(p-1)$? We can show: $h_p(2) \lesssim 2.5p$.

* We showed: $f(\mathbb{Z}_p^k) = O(\log k \cdot h_p(k))$.

Open problems

* We know: $h_p(k) \geq k(p-1)$ and showed: $h_p(k) \leq \binom{k+1}{2}(p-1)$.

  Is $h_p(k) = k(p-1)$?

  In particular, is $h_p(2) = 2(p-1)$? We can show: $h_p(2) \lesssim 2.5p$.

* We showed: $f(\mathbb{Z}_p^k) = O(\log k \cdot h_p(k))$.

  Is it true that $f(\mathbb{Z}_p^k) = O(h_p(k))$?

Open problems

* We know: $h_p(k) \geq k(p-1)$ and showed: $h_p(k) \leq \binom{k+1}{2}(p-1)$.

  Is $h_p(k) = k(p-1)$?

  In particular, is $h_p(2) = 2(p-1)$? We can show: $h_p(2) \lesssim 2.5p$.

* We showed: $f(\mathbb{Z}_p^k) = O(\log k \cdot h_p(k))$.

  Is it true that $f(\mathbb{Z}_p^k) = O(h_p(k))$?

* What is $f(A)$ for a general abelian group?

Open problems

* We know: $h_p(k) \geq k(p-1)$ and showed: $h_p(k) \leq \binom{k+1}{2}(p-1)$.
  Is $h_p(k) = k(p-1)$?
  In particular, is $h_p(2) = 2(p-1)$? We can show: $h_p(2) \lesssim 2.5p$.

* We showed: $f(\mathbb{Z}_p^k) = O(\log k \cdot h_p(k))$.
  Is it true that $f(\mathbb{Z}_p^k) = O(h_p(k))$?

* What is $f(A)$ for a general abelian group?

Thank you for listening!